ESS100 Modelling and simulation
Solution to exam Tue, 14 December 2004

Exercise 1

(a) The implicit numerical methods have a larger stability region compared with explicit methods. Unfortunately, the implicit methods are more complex to solve.

(b) An OE-model can not be written as a linear regresion $\hat{y} = \theta^T \phi(t)$, because $\phi(t)$ for an OE-model will include old model output signals and these a dependent on the parameter vector θ. For OE-models an iterative Gauss-Newton method is necessary.

(c) System is on standard form:

$$\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} \dot{x} + \begin{bmatrix} -A & 0 \\ 0 & I \end{bmatrix} x = \begin{bmatrix} B \\ D \end{bmatrix} u$$

The smallest integer k for which $N^k = 0$, is called the *index*. For our system *index*=2, because $N^1 \neq 0$ but $N^2 = 0$.

(d) Stiff differential equations are characterized by the fact that their solutions include both fast and slow components, this usually leads to problem during simulation (time-consuming).

(e) There exist more developed methods for linear that nonlinear systems, tex frequency analysis, poles-zeros etc. The linearized model is only valid in a region closed to the stationary point around which the system has been linearized, not a global model.

Exercise 2

For the closed loop system y and u can be expressed as:

$$y = \frac{1}{1 + FG}(Gv + He), \quad u = \frac{1}{1 + FG}(v - FHe)$$

or fourier transformed

$$Y(\omega) = \frac{1}{1 + F(i\omega)G(i\omega)}(G(i\omega)V(\omega) + H(i\omega)E(\omega)),$$

$$U(\omega) = \frac{1}{1 + F(i\omega)G(i\omega)}(V(\omega) - F(i\omega)H(i\omega)E(\omega))$$

The spectrum for u can than be calculated as

$$\Phi_u(\omega) = |U(\omega)|^2 = U(\omega)\overline{U(\omega)} = \frac{1}{|1 + F(i\omega)G(i\omega)|^2}(\Phi_v(\omega) + |F(i\omega)H(i\omega)|^2\Phi_e(\omega))$$

and the cross spectrum can be calculated in a similar way

$$\Phi_{yu}(\omega) = Y(\omega)\overline{U(\omega)} = \frac{1}{|1 + F(i\omega)G(i\omega)|^2}(G(i\omega)\Phi_v(\omega) - F(i\omega)|H(i\omega)|^2\Phi_e(\omega))$$

The estimation of $G(p)$ can then given as

$$\hat{G}(i\omega) = \frac{\Phi_{yu}(\omega)}{\Phi_u(\omega)} = \frac{G(i\omega)\Phi_v(\omega) - F(i\omega)|H(i\omega)|^2\Phi_e(\omega)}{\Phi_v(\omega) + |F(i\omega)H(i\omega)|^2\Phi_e(\omega)}$$

If $v(t) \approx 0$ the spectrum for v will be 0, i.e. $\Phi_v(\omega) = 0$ and the estimation of $G(p)$ then becomes

$$\hat{G}(i\omega) = \frac{-1}{F(i\omega)}$$

i.e. the estimation of $G(p)$ is not correct. This verifies that for system in closed loop spectral analysis does not work.

Exercise 3

Introduce F as state variable x_3. The dynamics for x_3 is according to the instruction

$$\dot{x}_3 = -\frac{1}{T}x_3 + \frac{1}{T}u$$

The dynamics for the fan process becomes:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\tfrac{g}{r+L}sinx_1 - \tfrac{k}{m}x_2 + \tfrac{r}{m(r+L)^2}x_3cosx_1 \\
\dot{x}_3 &= -\tfrac{1}{T}x_3 + \tfrac{1}{T}u
\end{aligned}$$

or

$$\dot{x} = f(x, u)$$

Linearize around the point $(x_0 = 0, u_0 = 0)$:

$$\Delta\dot{x} = \left.\frac{\partial f}{\partial x}\right|_{x_0,u_0} \Delta x + \left.\frac{\partial f}{\partial u}\right|_{x_0,u_0} \Delta u = A\Delta x + B\Delta u$$

where

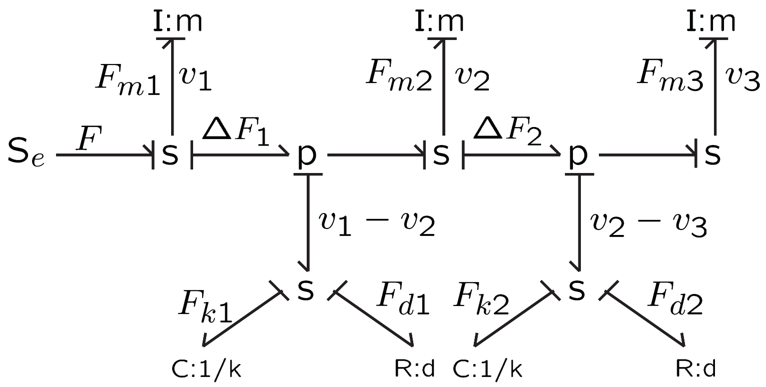
$$\left.\frac{\partial f}{\partial x}\right|_{x_0,u_0} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{g}{r+L} & -\frac{k}{m} & \frac{r}{m(r+L)^2} \\ 0 & 0 & -\frac{1}{T} \end{bmatrix}, \quad \left.\frac{\partial f}{\partial u}\right|_{x_0,u_0} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{T} \end{bmatrix}$$

The linear model then becomes:

$$\Delta\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{g}{r+L} & -\frac{k}{m} & \frac{r}{m(r+L)^2} \\ 0 & 0 & -\frac{1}{T} \end{bmatrix} \Delta x + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{T} \end{bmatrix} \Delta u$$

Exercise 4

(a) Bond graph according to figure:



(b) Choose state-variables as the flow variables at the I-elements and effort variables at the C-elements. From the bond graph five state variables can be found, v_1, v_2, v_3, F_{k1} and F_{k2}.

$$\begin{aligned}
\dot{v}_1 &= \tfrac{1}{m}F_{m1} = \tfrac{1}{m}(F - \Delta F_1) = \tfrac{1}{m}(F - F_{k1} - F_{d1}) = \tfrac{1}{m}(F - F_{k1} - d(v_1 - v_2)) \\
\dot{v}_2 &= \tfrac{1}{m}F_{m2} = \tfrac{1}{m}(\Delta F_1 - \Delta F_2) = \tfrac{1}{m}(F_{k1} + d(v_1 - v_2) - F_{k2} - d(v_2 - v_3)) \\
\dot{v}_3 &= \tfrac{1}{m}F_{m3} = \tfrac{1}{m}\Delta F_2 = \tfrac{1}{m}(F_{k2} + d(v_2 - v_3)) \\
\dot{F}_{k1} &= k(v_1 - v_2) \\
\dot{F}_{k2} &= k(v_2 - v_3)
\end{aligned}$$

The output signal can be derived as $L_0 + F_{k1}/k + F_{k2}/k$. The state space model becomes:

$$\dot{x} = \begin{bmatrix} -d/m & d/m & 0 & -1/m & 0 \\ d/m & -2d/m & d/m & 1/m & -1/m \\ 0 & d/m & -d/m & 0 & 1/m \\ k & -k & 0 & 0 & 0 \\ 0 & k & -k & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1/m \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = L_0 + \begin{bmatrix} 0 & 0 & 0 & 1/k & 1/k \end{bmatrix} x$$

(c) Look at the structure above and augment for n wagons.

Exercise 5

(a) Introduce a state variable at each integrator, in this case three states, x_1, x_2 and x_3, on the right hand side of each integrator we have the state variables and on the left hand side we have the time derivative of the state variables. From the Simulink scheme the derivatives can now be calculated as:

$$\begin{aligned} \dot{x}_1 &= u - 5x_1 + 2x_2 + x_3 \\ \dot{x}_2 &= u - 0.5x_2 + x_3 \\ \dot{x}_3 &= u - 200x_3 \end{aligned}$$

the output signal is given as the sum of the state variables, i.e.

$$y = x_1 + x_2 + x_3 = [1\ 1\ 1]x$$

on state space form the model can be written as

$$\dot{x} = \begin{bmatrix} -5 & 2 & 1 \\ 0 & -0.5 & 1 \\ 0 & 0 & -200 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = [1\ 1\ 1]x$$

(b) Step response b belongs to the Simulink-model. The system has only real poles (A-matrix on diagonal form =¿ diagonal elements = eigenvalues) this the system non-oscillating, i.e. not response a. The slowest mode in the system will determine the transient, approximative time constant $T = 2s$ $(T = 1/0.5)$. From the figure step response b can now be identified, because response c has a time constant of approximately 5s.

(c) x_3 corresponds to the fastest dynamics and is approximately 50 times faster than the other dynamics. Approximate with a static relationship $200x_3 = u$. Replace x_3 with $u/200$ in the model.

The simplified model becomes

$$\dot{x} = \begin{bmatrix} -5 & 2 \\ 0 & -0.5 \end{bmatrix} x + \begin{bmatrix} 201/200 \\ 201/200 \end{bmatrix} u$$

$$y = [1\ 1]x + 1/200u$$