CHALMERS UNIVERSITY OF TECHNOLOGY
Department of Signals and Systems
Division of Automatic Control, Automation and Mechatronics

EXAMINATION IN NONLINEAR AND ADAPTIVE CONTROL

(Course ESS076)

Monday January 14, 2013

Time and place:	14:00 - 18:00 at Väg och vatten
Teacher:	Torsten Wik (5146 or 0739 870570)

The following items are allowed (controlled by teacher):

1. *Control Theory* (Glad Ljung) or *Applied Nonlinear Control* (Slotine/Li)
2. ESS076 Supplement
3. Mathematical handbooks of tables such as Beta Mathematics Handbook.
4. Course summary from Lecture 18

Notes, calculator, mobile telephones, laptops or palmtops, are not allowed! Reasonable notes in the textbook are allowed but no solved problems.

The total points achievable are 30 with the following scales for grading

Grade 3:	at least 12 points
Grade 4:	at least 18 points
Grade 5:	at least 24 points

Incorrect solutions with significant errors, unrealistic results or solutions that are difficult to follow generally result in 0 points.

Grading results are posted not later than January 28. Review of the grading is offered on Tuesday January 29 at 11:45-12:30. If you cannot attend on this occasion, any objections concerning the grading must be filed in written form not later than two weeks after the regular review occasion.

Good Luck!

1. Consider the system

$$\begin{aligned}\dot{x} &= -x^3 + y^2 \\ \dot{y} &= -2xy - y\end{aligned}$$

   (a) Can we conclude stability by investigating the local properties around the equilibrium point?

   1 p.

   (b) Show that the system is globally asymptotically stable.

   2 p.

2. Determine a state feedback that exactly linearizes the system

$$\begin{aligned}\dot{x}_1 &= -3x_2 \\ \dot{x}_2 &= x_1 - x_2 + x_2^2 - (2 + \cos x_2)u\end{aligned}$$

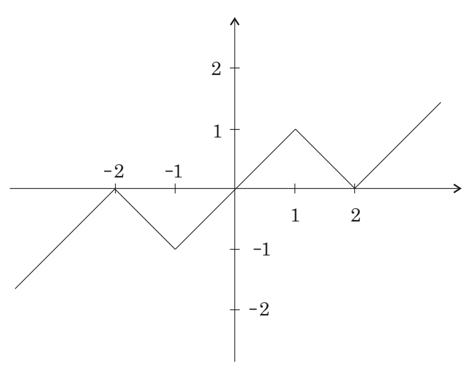
   and gives the poles -1 and -3 in the transfer function from reference to the output signal x_2.

   2 p.

3. A system described by the transfer function

$$G(s) = \frac{K}{(1+sT)^3}$$

   is connected in negative feedback with the nonlinearity shown in the figure below. What are the stability constraints on the gain K according to the small gain theorem?



   2 p.

4. A mass is moving around an equilibrium governed by an external force u, a linear spring and viscous friction according to

$$\ddot{z} + \dot{z} + \Phi(\dot{z}) + z = u$$

where $0.1v^2 \leq \Phi(v)v \leq v^2$

(Note: All subproblems can be solved independently of eachother!)

(a) Prove that $\Phi(\cdot)$ is strictly passive

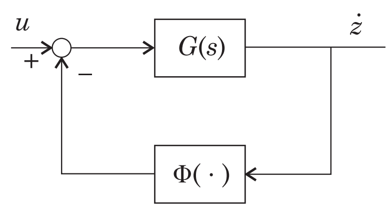
1 p.

(b) Show that the system can be written as a feedback interconnection of a linear system

$$G(s) = \frac{s}{s^2 + s + 1}$$

and the static nonlinearity $\Phi(\cdot)$ as shown in the block diagram below.

2 p.



(c) Show that the linear system G is strictly passive.

2 p.

(d) Is the origin ($\dot{z} = z = u = 0$) asymptotically stable (motivate)?

1 p.

5. A nonlinear servo system is described by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -2x_2 - f(x_1) \end{aligned}$$

The nonlinearity should obey $f(x) = x$ but in reality it does not. Use the circle criterion to determine a constraint of the type

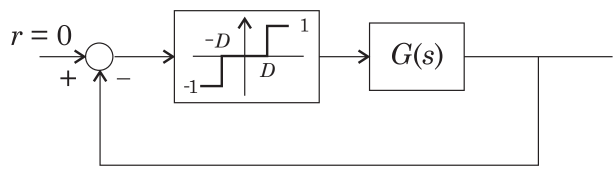
$$k_1 \leq \frac{f(x)}{x} \leq k_2$$

for which the system is stable.

4 p.

6. A well insulated room is heated with an electric radiator having a thermostat acting as an ideal relay with a deadzone. The system can be regarded as the feedback system in the figure below, where the linear part is

$$G(s) = \frac{K}{s(1+5s)^2}$$



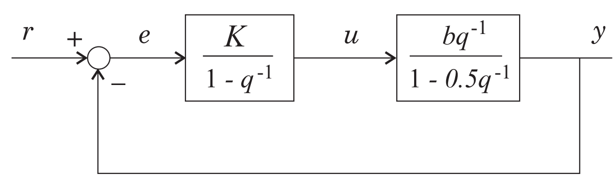
The deadzone parameter D is a design parameter for the thermostat. If it is too small the system will oscillate. Approximately how large does D have to be to avoid self-oscillation.

5 p.

7. A first order process with varying gain is to be controlled using a discrete time integral controller (see the figure). Determine a RLS based indirect STR aiming for a double pole in α for the closed loop system.

Because there are not enough degrees of freedom α cannot be chosen arbitrarily with this approach. What should α be?

4 p.



8. In a factory trolleys are transporting goods on a rail, with a friction coefficient d. If m is the mass of the goods and the trolley the movement is given by

$$m\ddot{y} = -d\dot{y} + u$$

where y is the position of the trolley and u is the applied force. In order to keep a high flow of goods we want the trolley to go from rest to a high velocity in a precalculated time t_f. At the same time we want to use as little energy as possible.

Assuming $m = 1$, $d = 1$, $t_f = 1$ and the desired final velocity $\dot{y}(t_f) = 1$, we can formulate this as an optimal control problem

$$\min_u \int_0^1 u^2(t)dt$$

subject to

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ x_1(0) &= 0, \quad x_2(0) = 0 \\ x_2(1) &= 1 \end{aligned}$$

where $x = \begin{bmatrix} y & \dot{y} \end{bmatrix}^T$. Determine the optimal $u(t)$, $0 < t < 1$.

4 p.

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1 a) $\left.\begin{array}{l}\dot{x}=0\\ \dot{y}=0\end{array}\right\} \Rightarrow (0,0)$ only eq. pt.

Linearization gives

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}}_{A}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\det(\lambda I - A) = \det\begin{bmatrix} \lambda & 0 \\ 0 & \lambda+1 \end{bmatrix} = \lambda(\lambda+1) = 0$$

$\Rightarrow \lambda_1 = 0 \ , \ \lambda_2 = -1$

$\Rightarrow$ cannot conclude stability

b) $V = \frac{1}{2}(\alpha x^2 + y^2) \ , \quad \alpha > 0$

$$\dot{V} = \alpha x(-x^3 + y^2) + y(-2xy - y)$$

$$= -\alpha x^4 - y^2 + \alpha x y^2 - 2xy^2$$

$$< 0 \quad \forall (x,y) \neq (0,0) \text{ if } \alpha = 2$$

Since $V(0) = 0$, $V > 0$, $V \to \infty$ as $(x,y) \to \infty$

the system is globally as. stable

2) $$u = \frac{1}{2 + \cos x_2}(x_2^2 + a x_2 - r)$$

gives
$$\dot{x}_1 = -3x_2$$
$$\dot{x}_2 = x_1 - x_2 + x_2^2 - (x_2^2 + a x_2 - r)$$

$$\Rightarrow \dot{x} = \underbrace{\begin{bmatrix} 0 & -3 \\ 1 & -(1+a) \end{bmatrix}}_{A} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$$

$$\det(\lambda I - A) = \lambda(\lambda + 1 + a) + 3 = 0$$

$$\lambda^2 + (1+a)\lambda + 3 = 0$$

$$\left(\lambda + \frac{1+a}{2}\right)^2 = -3 + \frac{(1+a)^2}{4}$$

$$\lambda = -\frac{1+a}{2} \pm \sqrt{-3 + \frac{(1+a)^2}{4}}$$

$$\Rightarrow \lambda = -1, -3 \quad \text{if } a = 3$$

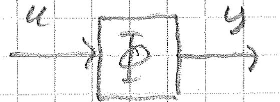
3) For the nonlinearity we have

$$|f(x)| \leq |x| \Rightarrow \|f\| \leq 1$$

$$\|G\| = \sup_{\omega} |G(i\omega)| = K$$

SGT gives stable if $\|f\|\|G\| = K < 1$

4 a)



$y = \Phi(u)$

Φ memoryless $\Rightarrow$ no state

$\Rightarrow$ output strictly passive $\Rightarrow$ strictly passive

$uy - \delta y^2 = u\Phi(u) - \delta\Phi^2(u) \geq 0$

$u\Phi(u) \geq 0.1\,u^2$

$(u\Phi(u))^2 \leq u^4 \quad \Rightarrow \Phi^2(u) \leq u^2$

$\Rightarrow uy - \delta y^2 \geq 0 \quad$ if $\quad 0 < \delta < 0.1$

b) $\quad \ddot{z} + \dot{z} + \Phi(\dot{z}) + z = u$

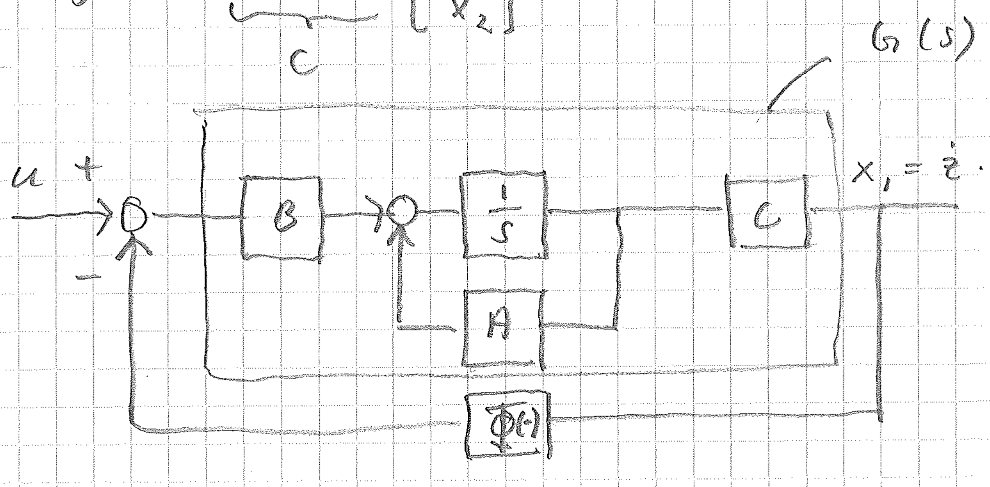
Let $[x_1 \ x_2] = [\dot{z} \ z]$

$\dot{x}_1 = -x_1 - \Phi(x_1) - x_2 + u$

$\dot{x}_2 = x_1$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} (u - \Phi(y))$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



$$G(s) = C[sI - A]^{-1}B = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s+1 & 1 \\ -1 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{s(s+1)+1} \begin{bmatrix} s & -1 \\ 1 & s+1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{s}{s^2+s+1}$$

c) $s^2+s+1 = 0 \Rightarrow s = \frac{1}{2}(-1 \pm j\sqrt{3}) \in LHP$

$$G(j\omega) = \frac{j\omega}{-\omega^2 + j\omega + 1} = \frac{j\omega(1-\omega^2-j\omega)}{(1-\omega^2)^2+\omega^2}$$

$$= \frac{\omega^2 + j\omega(1-\omega^2)}{(1-\omega^2)^2+\omega^2}$$

$\Rightarrow Re\{G(j\omega)\} > 0 \quad \forall \omega > 0$

$$\lim_{\omega \to \infty} \omega^2 Re\{G(j\omega)\} = \lim_{\omega \to \infty} \frac{\omega^4}{(1-\omega^2)^2+\omega^2} = 1$$

$\therefore \lim_{\omega \to \infty} \omega^2 Re\{G(j\omega)\} > 0$

$\Rightarrow$ G is strictly positive real according to Definition 3 $\Rightarrow$ strictly passive

d) G and Φ strictly passive, Theorem 1.1 (supplement) $\Big\}$ asymptotically stable

5) $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -2x_2 - f(x_1) \end{cases}$

$$\Leftrightarrow \quad \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (-f(y))$$

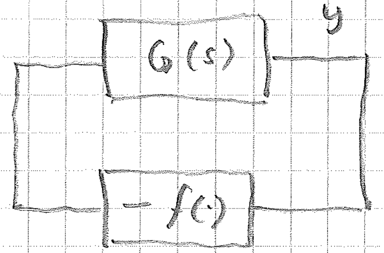
$$y = [1 \quad 0] x$$

$$G(s) = [1 \quad 0] \begin{bmatrix} s & -1 \\ 0 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= [1 \quad 0] \frac{1}{s(s+2)} \begin{bmatrix} s+2 & 1 \\ 0 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

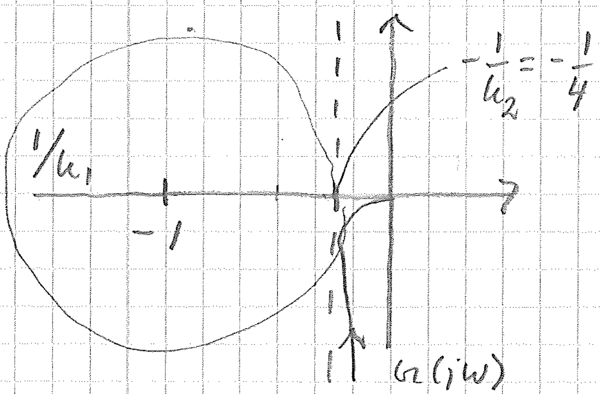
$$= \frac{1}{s(s+2)}$$

$\Leftrightarrow$



$$G(j\omega) = \frac{1}{j\omega(j\omega+2)} = \frac{-j\omega(2-j\omega)}{\omega^2(\omega^2+4)}$$

$$Re\{G(j\omega)\} = \frac{-1}{\omega^2+4} > -\frac{1}{4}$$



$G(j\omega)$ will not enter the disc (k_1, k_2) if $k_2 < 4$ and $k_1 = 0$

6)

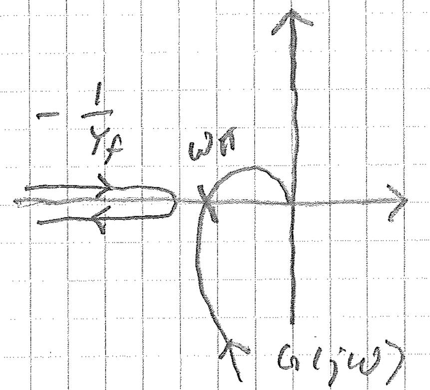
The describing function for the relay is (from textbook)

$$Y_f(C) = \frac{4}{\pi C}\sqrt{1 - D^2/C^2}, \quad C > D$$

This means that for a given D

$$-\frac{1}{Y_f(C)} = \frac{-\pi C^2}{4\sqrt{C^2 - D^2}} \to \begin{cases} -\infty, & C \to D \\ -\infty, & C \to \infty \end{cases}$$

Hence there is a point on the negative real axis where $-\frac{1}{Y_f(C)}$ turns.



To avoid oscillations $G(j\omega)$ should cross the negative real axis to the right of that point

$$\frac{d}{dC}\left\{\frac{1}{Y_f}\right\} = \frac{\pi}{4(C^2 - D^2)}\left(2C\sqrt{C^2 - D^2} - C^2\frac{2C}{2\sqrt{C^2 - D^2}}\right) = 0$$

$$\Rightarrow \quad 4(C^2 - D^2) - 2C^2 = 0$$

$$2C^2 - 4D^2 = 0 \Rightarrow C^2 = 2D^2$$

$$\angle G(j\omega_\pi) = -\frac{\pi}{2} - 2\,\text{atan}\, 5\omega_\pi = -\pi$$

$$\Rightarrow \quad \omega_\pi = \frac{1}{5}$$

$$|G(j\omega_\pi)| = \frac{K}{2\omega_\pi} < \frac{1}{Y_f(C)} = \frac{2\pi D^2}{4\sqrt{2D^2 - D^2}}$$

$$= \frac{\pi D}{2}$$

$$\Rightarrow \quad D > \frac{K}{\pi \omega_\pi} = \frac{5K}{\pi}$$

7) Loop gain $L(q) = \dfrac{K_r b\, q^{-1}}{(1-q^{-1})(1-0.5q^{-1})} = \dfrac{K_r b\, q^{-1}}{1-1.5q^{-1}+0.5q^{-2}}$

Closed loop

$$y(t) = \frac{L}{1+L}\, r(t) = \frac{K_r b\, q^{-1}}{1+(Kb-1.5)q^{-1}+0.5q^{-2}}$$

Double pole in $q = \alpha$

$$1 + (Kb-1.5)q^{-1} + 0.5q^{-2} = (1 - q^{-1}\alpha)^2$$

$$= 1 - 2\alpha q^{-1} + \alpha^2 q^{-2}$$

$$\Rightarrow \left.\begin{aligned} K_r b - 1.5 &= -2\alpha \\ 0.5 &= \alpha^2 \end{aligned}\right\} \Rightarrow \alpha = \frac{1}{\sqrt{2}} \quad \text{(not tunable)}$$

$$K_r = \frac{3 - 2\sqrt{2}}{2b} \quad \text{(b known)}$$

Estimate $\theta = b$

$$y(t) = \frac{b q^{-1}}{1 - 0.5q^{-1}}\, u(t)$$

$$y(t) - 0.5y(t-1) = b\,u(t-1)$$

Let $z(t) = y(t) - 0.5\,y(t-1) = \varphi^T(t)\theta$

where $\varphi^T(t) = u(t)$, $\theta = b$

RLS then gives

1. $\hat{b}(t) = \hat{b}(t-1) + K(t)\varepsilon(t)$
2. $\varepsilon(t) = z(t) - \hat{b}(t-1)u(t-1)$
3. $K(t) = P(t)u(t-1)$
4. $P(t) = \dfrac{1}{\lambda}\left(P(t-1) - \dfrac{P^2(t-1)\,u^2(t-1)}{\lambda + P(t-1)u^2(t-1)}\right)$

$$= \frac{P(t-1)}{\lambda + P(t-1)u^2(t-1)}$$

Run the above steps from 4 to 1 and then let

$$K_r = \frac{3-2\sqrt{2}}{2\hat{b}(t)}$$

and then calculate

$$u(t) = u(t-1) + K\left(y(t) - r(t)\right)$$

8) End condition, fixed final time $t_f = 1$

Hamiltonian

$$H = n_0 u^2 + \lambda^T (Ax + Bu)$$

Abnormal case $n_0 = 0 \Rightarrow \frac{\partial H}{\partial u} = \lambda^T B$

independent of u and, hence, there would be no optimum

Normal case $n_0 = 1$

$$\frac{\partial H}{\partial u} = 2u + \lambda^T B = 0 \Leftrightarrow u = -\frac{1}{2}\lambda^T B$$

The adjoint eqn

$$\dot{\lambda} = -\frac{\partial H}{\partial x}^T = -A^T\lambda = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

End condition $\psi = x_2 - 1 = 0$, $t = t_f$

$$\lambda(t_f) = \psi_x^T \mu = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\mu$$

$$\begin{cases} \dot{\lambda}_1 = 0, \ \lambda_1(1) = 0 \Rightarrow \lambda_1 \equiv 0 \Rightarrow \\ \dot{\lambda}_2 = \lambda_2, \ \lambda_2(1) = \mu \Rightarrow \lambda_2 = \mu e^{t-1} \end{cases}$$

$$\Rightarrow u = -\frac{1}{2}\mu e^{t-1}$$

Determine μ!

$\ddot{y} = -\dot{y} + u$ can be written

$$\dot{x}_2 + x_2 = -\frac{1}{2} e^{t-1} \mu$$

$$\frac{d}{dt}(x_2 e^t) = -\frac{1}{2} e^{2t-1} \mu$$

$$x_2 e^t = -\frac{1}{4} e^{2t-1} \mu + c \quad , \quad c \text{ const.}$$

$$x_2 = -\frac{1}{4} e^{t-1} \mu + c e^{-t}$$

$$x_2(0) = -\mu \frac{e^{-1}}{4} + c = 0 \Rightarrow c = \mu \frac{e^{-1}}{4}$$

$$x_2(1) = -\mu \frac{1}{4} + c e^{-1} = 1$$

$$\Rightarrow \mu \left(-\frac{1}{4} + \frac{e^{-2}}{4} \right) = \frac{\mu}{4}(e^{-2} - 1) = 1$$

$$\Rightarrow \mu = \frac{4}{e^{-2} - 1}$$