CHALMERS UNIVERSITY OF TECHNOLOGY
Department of Signals and Systems
Control, Automation and Mechatronics

ESS 076 Nonlinear and Adaptive Control
Final exam 2010-05-26

V 14.00 – 18.00

Teacher: Bo Egardt, tel 3721.

The following items are allowed to bring to the exam:

- The course textbook (Glad/Ljung: Control Theory or Slotine/Li: Applied Nonlinear Control).
- ESS076 Supplement.
- Beta.

Grading: The exam consists of 5 problems of in total 30 points. The nominal grading is 12 (3), 18 (4) and 24 (5). Solutions may be short, but should always be clear and well motivated!
Grading results are posted not later than June 10 at the billboard on the 5th floor. Review of the grading is offered on June 10 at 12.30 – 13.30. If you cannot attend at this occasion, any objections concerning the grading must be filed in written form not later than two weeks after the regular review occasion.

Note that solutions should be given in English!

GOOD LUCK!

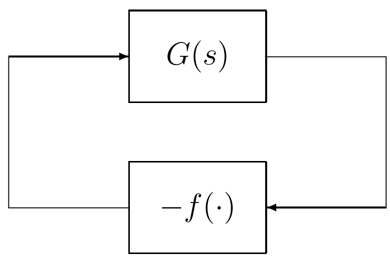
Problem 1.

a. By using a quadratic Lyapunov function, show that the origin is an asymptotically stable stationary point of the system

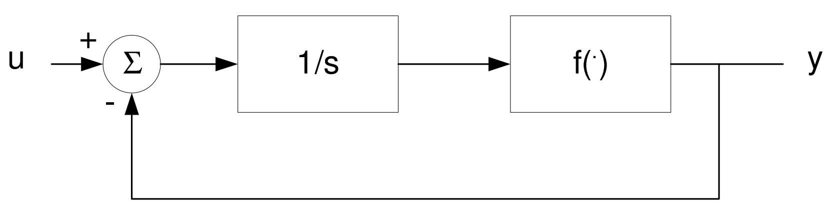
$$\dot{x}_1 = 2x_2$$
$$\dot{x}_2 = -x_1 - x_2(1 - x_1^2)$$

and estimate its region of attraction. (2 p)

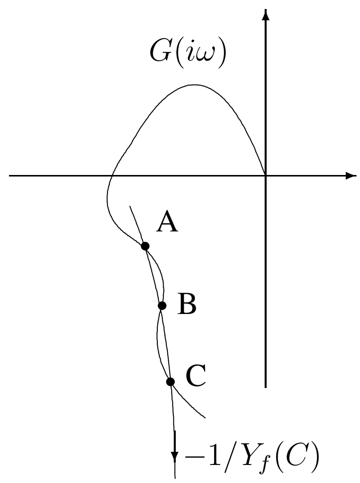
b. An LTI system with transfer function $G(s)$ is connected in feedback with a static nonlinearity, as depicted below. Give a sufficient condition on the nonlinearity $f(\cdot)$ for the closed-loop system to be input-output stable, if $G(s)$ fulfills $\text{Re}(G(i\omega)) > -1$. (2 p)



c. Show that the system depicted below, with input u and output y, and with the nonlinearity f in the sector $[0, \infty]$, is passive. (2 p)



d. A linear system with the transfer function $G(s)$ is connected in feedback with the nonlinearity $-f(\cdot)$. The figure below shows the Nyquist curve $G(i\omega)$ and the curve $-1/Y_f(C)$, where $Y_f(C)$ is the describing function of the nonlinearity. What can be deduced from the figure in terms of possible self-oscillations and their stability? Give your conclusions using the letter markings in the figure. (2 p)



e. Determine a control law for the system

$$
\begin{aligned}
\dot{x}_1 &= e^{x_2} - 1 \\
\dot{x}_2 &= u \\
y &= x_1
\end{aligned}
$$

so that the relation between reference signal and output becomes

$$\ddot{y} + \dot{y} + y = r$$

(2 p)

Solution:

a. $V = \frac{1}{2}x_1^2 + x_2^2$ *gives* $\dot{V} = -2x_2^2(1 - x_1^2) \leq 0$ *for* $|x_1| \leq 1$. *According to LaSalle's theorem, solutions starting in the set* $\{x|V(x) < 1/2\}$ *converges to the set where* $\dot{V} \equiv 0$, *but* $x_2 \equiv 0$ *implies* $x_1 \equiv 0$ *from the system equations. Hence, the origin is a.s. with region of attraction* $\{x|V(x) < 1/2\}$ *(the interior of an ellipsoid).*

b. *Apply the circle criterion with* $k_1 = 0$ *and* $k_2 = 1$. *The system is I/O-stable if* $f(0) = 0$ *and* $0 \le f(x)/x \le 1,\ x \ne 0$.

c. *Using* $V(x) = \int^x f(\sigma)d\sigma$ *(which is nonnegative since* $f \in [0, \infty]$*) gives* $\dot{V} = f(x) \cdot (u - y) = u \cdot y - y^2 \le u \cdot y$.

d. A *and* C *stable oscillations,* B *unstable oscillation. This conclusion is drawn by comparing with figures or/and arguments in the course book.*

e. $\dot{y} = e^{x_2} - 1$ *and* $\ddot{y} = e^{x_2}u$ *suggests the control law* $u = -1 + e^{-x_2}(1 - x_1) + e^{-x_2}r$.

Problem 2.
Consider the system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_1 - \text{sat}(2x_1 + x_2).$$

where sat$(\cdot)$ is the saturation function,

$$\text{sat}(x) = \begin{cases} -1, & x < -1 \\ x, & -1 \le x \le 1 \\ 1, & x > 1 \end{cases}$$

a. Show that the origin is asymptotically stable. (2 p)

b. Show that the origin is *not* globally asymptotically stable. (3 p)

Solution:

a. Linearization around the origin gives eigenvalues with negative real values, so the origin is asymptotically stable by Lyapunov-Poincaré.

b. The system has another stationary point in $x_1 = 1$, $x_2 = 0$. A trajectory starting in that point never reaches the origin, so the origin is not globally asymptotically stable. Another way to prove that trajectories exist that never reach the origin is to use the boundary $x_1x_2 = c$ for a large constant c, or for example a boundary defined by the lines $x_1 = a$ and $x_2 = b$ for large values of a and b (draw a figure and indicate the direction of $\dot{x}$ on the boundary).

Problem 3.
Consider the nonlinear system

$$\dot{x}_1 = \sin x_2 + x_3$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = u$$
$$y = x_1$$

a. Determine a nonlinear state feedback law that results in a linear input-output relation. (2 p)

b. Determine the zero dynamics. (1 p)

c. Suggest a redefinition of the system output that makes it possible to exactly linearize the system in *all* its states. (2 p)

Solution:

a. Differentiate to get $\dot{y} = \dot{x}_1 = \sin x_2 + x_3$ *and* $\ddot{y} = x_3 \cos x_2 + u$ *which suggests the control law* $u = r - x_3 \cos x_2$*, giving the closed-loop equation* $\ddot{y} = r$.

b. With new state variables $z_1 = y = x_1$ *and* $z_2 = \dot{y} = \sin x_2 + x_3$*, we get the closed-loop system*

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= r\end{aligned}$$

A one-to-one variable transformation can be obtained by defining e.g. $z_3 = x_2$*. Now, put* $r \equiv 0$ *and* $z_1(0) = z_2(0) = 0$ *to reveal the zero dynamics, giving* $\dot{z}_3 + \sin z_3 = 0$.

c. Choose e.g. $y = x_1 - x_2$*, leading to*

$$\begin{aligned}\dot{y} &= \sin x_2 \\ \ddot{y} &= x_3 \cos x_2 \\ \dddot{y} &= -x_3^2 \sin x_2 + u \cos x_2\end{aligned}$$

and the control law $u = \frac{1}{\cos x_2}(x_3^2 \sin x_2 + r)$ *gives* $\dddot{y} = r$.

Problem 4.
A certain disease is characterized by an increased amount of uric acid in the blood. The excess uric acid can be counteracted by a medicine that is injected at the rate u. A greatly simplified model, describing how the amount of uric acid x depends on the medicine taken, is given by

$$\dot{x} = 1 - x - u$$

Consider now the problem to determine a suitable injection rate of medicine to eliminate the uric acid. Assume $x(0) = 1$ and that the desired final state is $x(t_f) = 0$. It is desirable to reach the final state rapidly, but it is also desirable to limit the amount of medicine because of the risk of side-effects. Therefore, the following criterion should be minimized:

$$\int_0^{t_f} (k + u^2(t))dt$$

Here, $k > 0$ is a given constant. Determine the optimal rate of injected medicine, and the corresponding final time t_f! (5 p)

Solution: *This is an optimal control problem with* $L = k+u^2$, $\Phi = 0$, $\Psi(x(t_f)) = x(t_f) = 0$. *Minimization of the Hamiltonian* $H = k + u^2 + \lambda(1 - x - u)$ *gives* $u = \lambda/2$. *The adjoint equation is given by*

$$\dot{\lambda} = -H_x^T = \lambda; \qquad \lambda(t_f) = \mu$$

and has the solution $\lambda(t) = Ce^t$ *for some* C. *This implies* $u = \frac{C}{2}e^t$ *and the differential equation for* x *can be solved:*

$$x(t) = 1 + \frac{C}{4}(e^{-t} - e^t)$$

Using the property $H \equiv 0$ *for the Hamiltonian, applied at* $t = 0$ *gives* $C = 2\sqrt{k}$, *and at* $t = t_f$ *gives the equation*

$$k + Ce^{t_f} - \frac{C^2}{4}e^{2t_f} = 0$$

which in turn gives the solution $e^{t_f} = \frac{1+\sqrt{1+k}}{\sqrt{k}}$.

Problem 5.
An integrating process with unknown (and in practice slowly varying) gain b,

$$G(s) = \frac{b}{s}$$

shall be controlled by a model reference adaptive controller. The desired closed-loop dynamics is given by the reference model

$$G_m(s) = \frac{b_m}{s + a_m}$$

a. Design an adaptive controller and give equations for the control law *and* the parameter adaptation rule. (3 p)

b. Show that the control error tends to zero for your adaptive controller, assuming b is constant. Which additional assumption(s) are needed? (2 p)

Solution:

a. The control law $u = -s_0 y + t_0 r$ gives the closed-loop system

$$y = \frac{bt_0}{p + bs_0} r$$

The stability based approach starts from the model

$$e = \frac{b}{p + a_m}[u + s_0^0 y - t_0^0 r],$$

where $t_0^0 = b_m/b$ and $s_0^0 = a_m/b$. With the control law $u = -s_0 y + t_0 r$ we get the error model

$$e = \frac{b}{p + a_m}[-\tilde{s}_0 y + \tilde{t}_0 r]$$

The parameter adjustment is now

$$\begin{bmatrix} \dot{s}_0 \\ \dot{t}_0 \end{bmatrix} = \gamma \begin{bmatrix} y \\ -r \end{bmatrix} e$$

b. Use the Lyapunov function $V = e^2 + \alpha(\tilde{s}_0{}^2 + \tilde{t}_0{}^2)$ and proceed as shown in the course supplement. We need to assume that the sign of b is known.