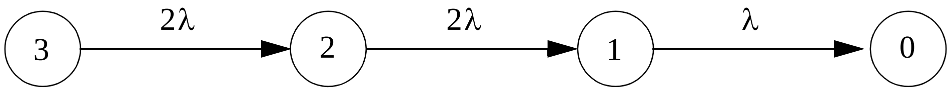
1.

a) We obtain the following Markov model



State label = number of operational nodes

which yields the following state transition rate matrix

$$\mathbf{Q} = \begin{bmatrix} -2\lambda & 2\lambda & 0 & 0 \\ 0 & -2\lambda & 2\lambda & 0 \\ 0 & 0 & -\lambda & \lambda \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and the following system of differential equations

$$\begin{cases} P'_3(t) = -2\lambda \cdot P_3(t) \\ P'_2(t) = 2\lambda \cdot P_3(t) - 2\lambda \cdot P_2(t) \\ P'_1(t) = 2\lambda \cdot P_2(t) - \lambda \cdot P_1(t) \\ P'_0(t) = \lambda \cdot P_1(t) \end{cases}$$

Assuming $\mathbf{P}(\mathbf{0}) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$ we solve the equation system using Laplace transforms

$$s \cdot \tilde{P}_3(s) - 1 = -2\lambda \cdot \tilde{P}_3(s)$$
$$s \cdot \tilde{P}_2(s) - 0 = 2\lambda \cdot \tilde{P}_3(s) - 2\lambda \cdot \tilde{P}_2(s)$$
$$s \cdot \tilde{P}_1(s) - 0 = 2\lambda \cdot \tilde{P}_2(s) - \lambda \cdot \tilde{P}_1(s)$$
$$s \cdot \tilde{P}_0(s) - 0 = \lambda \cdot \tilde{P}_1(s)$$

$$\tilde{P}_3(s) = \frac{1}{s + 2\lambda} \propto P_3(t) = e^{-2\lambda t}$$

$$\tilde{P}_2(s) = \frac{2\lambda \cdot \tilde{P}_3(s)}{s + 2\lambda} = \frac{2\lambda}{(s + 2\lambda)^2} \propto P_3(t) = 2\lambda t e^{-2\lambda t}$$

$$\tilde{P}_1(s) = \frac{2\lambda \cdot \tilde{P}_2(s)}{(s+\lambda)} = \frac{4\lambda^2}{(s+\lambda)(s+2\lambda)^2} = \frac{4}{s+\lambda} - \frac{4}{s+2\lambda} - \frac{4}{(s+2\lambda)^2}$$

$$P_1(t) = 4e^{-\lambda t} - 4e^{-2\lambda t} - 4\lambda t e^{-2\lambda t}$$

The reliability of the systemet is

$$R_{sys} = P_3(t) + P_2(t) + P_1(t) =$$
$$= 4e^{-\lambda t} - 3e^{-2\lambda t} - 2\lambda t e^{-2\lambda t}$$

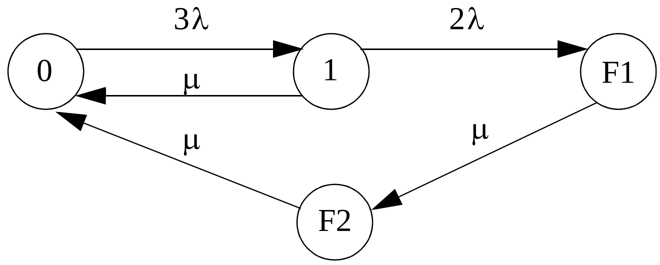
b) The MTTF of the system is

$$MTTF = \int_0^\infty R_{sys} dt$$
$$= \int_0^\infty 4e^{-\lambda t} dt - \int_0^\infty 3e^{-2\lambda t} dt - \int_0^\infty 2\lambda t e^{-2\lambda t} dt$$
$$= \frac{4}{\lambda} - \frac{3}{2\lambda} - \frac{1}{2\lambda} = \frac{2}{\lambda}$$

c) Let F denote the expected time to the first node failure.

$$F = \int_0^\infty P_3(t) \cdot dt$$
$$= \int_0^\infty e^{-2\lambda t} dt = \frac{1}{2\lambda}$$

2. The system can be modeled by the following state diagram.



States:

0: Three modules working.

1 : Two modules working, one module faulty.

F1: One module working, two modules working, system down

F2: Two modules working, one module faulty, system down

We obtain the following Q-matrix:

$$Q = \begin{bmatrix} -3\lambda & 3\lambda & 0 & 0 \\ \mu & -(2\lambda+\mu) & 2\lambda & 0 \\ 0 & 0 & -\mu & \mu \\ \mu & 0 & 0 & -\mu \end{bmatrix}$$

Steady-state solution:

$$\begin{aligned} \lim_{t\to\infty} P'_k(t) &= 0 \\ \lim_{t\to\infty} P_k(t) &= \Pi_k \end{aligned}$$

$$\begin{cases} 0 = -3\lambda\Pi_0 + \mu\Pi_1 + \mu\Pi_{F2} \\ 0 = 3\lambda\Pi_0 - (2\lambda+\mu)\Pi_1 \\ 0 = 2\lambda\Pi_1 - \mu\Pi_{F1} \\ 0 = \mu\Pi_{F1} - \mu\Pi_{F2} \end{cases}$$

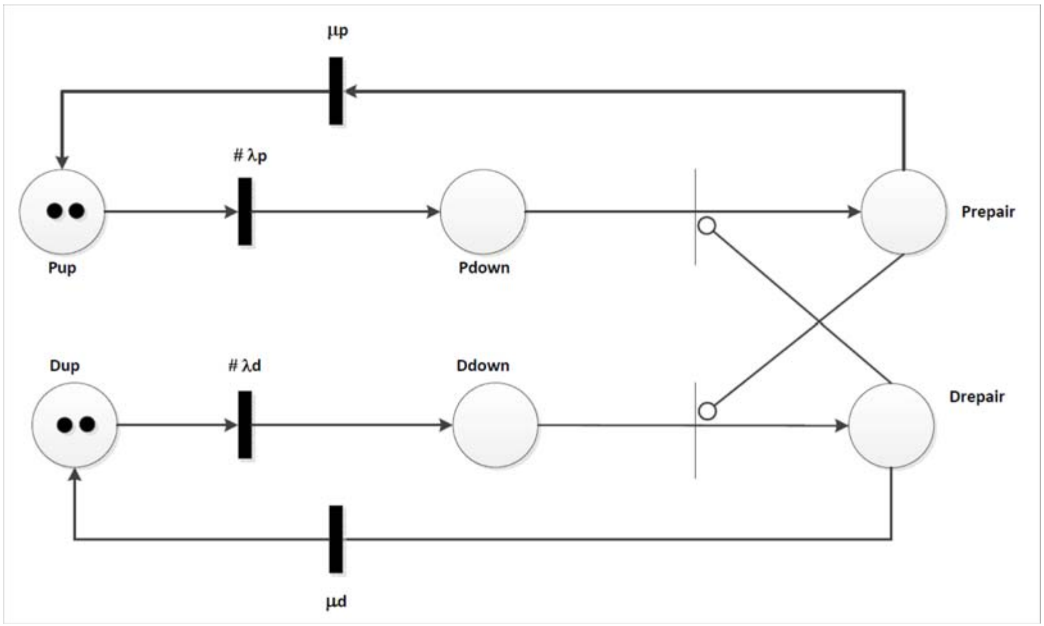
$$\begin{aligned} \Pi_{F1} &= \Pi_{F2} \\ \Pi_1 &= \frac{\mu}{2\lambda}\Pi_{F1} = \frac{\mu}{2\lambda}\Pi_{F2} \\ \Pi_0 &= \frac{2\lambda+\mu}{3\lambda}\Pi_1 = \frac{2\lambda+\mu}{3\lambda}\frac{\mu}{2\lambda}\Pi_{F2} \\ 1 &= \Pi_0 + \Pi_1 + \Pi_{F1} + \Pi_{F2} \\ &= \Pi_{F2}\left(\frac{2\lambda+\mu}{3\lambda}\frac{\mu}{2\lambda} + \frac{\mu}{2\lambda} + 1 + 1\right) \\ &= \Pi_{F2}\frac{2\lambda\mu + \mu^2 + 3\lambda\mu + 12\lambda^2}{6\lambda^2} \\ \Rightarrow \Pi_{F2} &= \frac{6\lambda^2}{5\lambda\mu + \mu^2 + 12\lambda^2} \end{aligned}$$

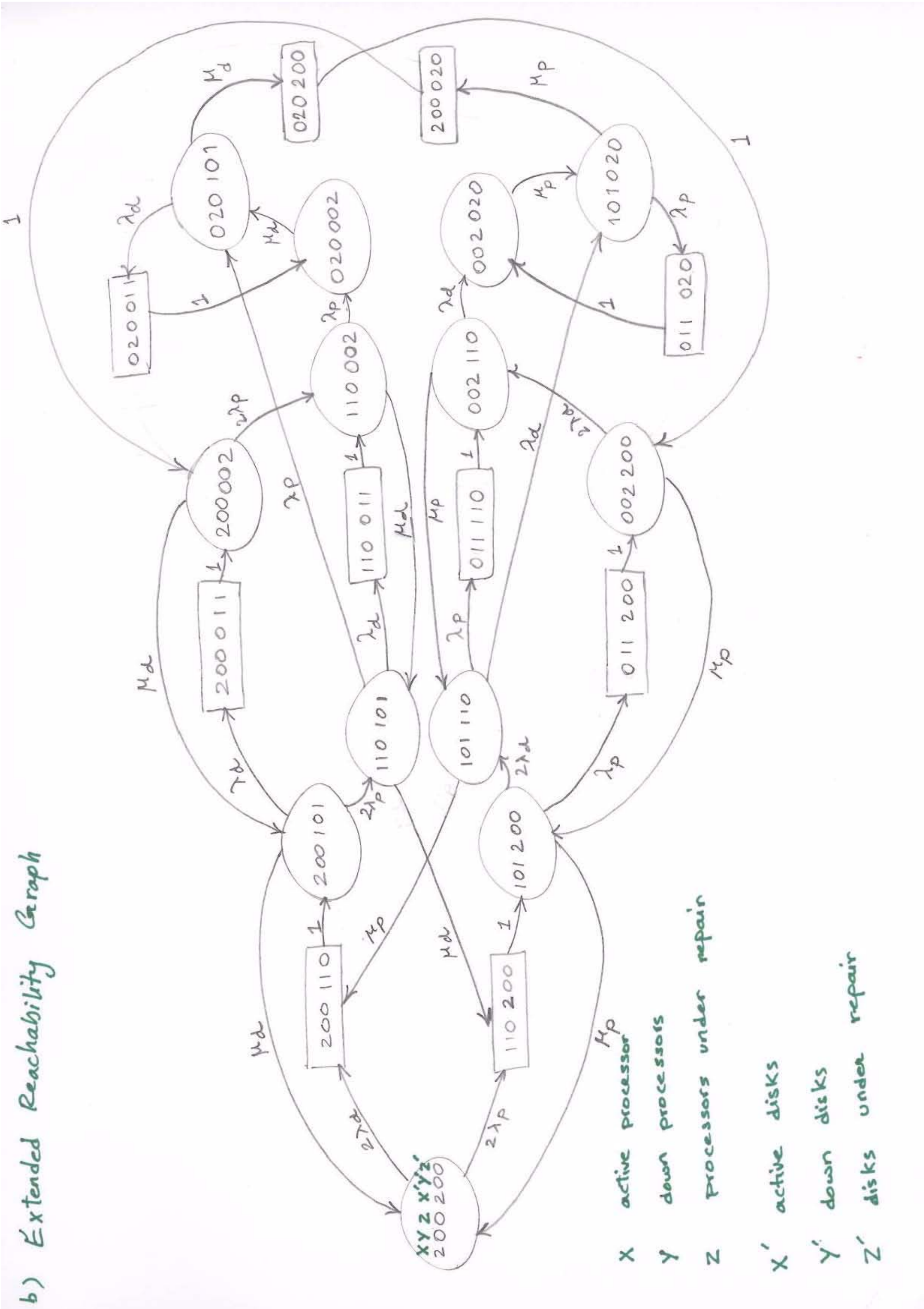
$$
\begin{aligned}
\Pi_0 &= \frac{2\lambda+\mu}{3\lambda}\frac{\mu}{2\lambda}\Pi_{F2} \\
&= \frac{2\lambda+\mu}{3\lambda}\frac{\mu}{2\lambda}\frac{6\lambda^2}{5\lambda\mu+\mu^2+12\lambda^2} \\
&= \frac{2\lambda\mu+\mu^2}{5\lambda\mu+\mu^2+12\lambda^2} \\
\Pi_1 &= \frac{\mu}{2\lambda}\frac{6\lambda^2}{5\lambda\mu+\mu^2+12\lambda^2} \\
&= \frac{3\lambda\mu}{5\lambda\mu+\mu^2+12\lambda^2}
\end{aligned}
$$

$$
\begin{aligned}
\lim_{t\to\infty} A(t) &= \Pi_0+\Pi_1 \\
&= \frac{5\lambda\mu+\mu^2}{5\lambda\mu+\mu^2+12\lambda^2}
\end{aligned}
$$

3\.

GSPN




b) Extended Reachability Graph
X active processor
Y down processors
Z processors under repair
X' active disks
Y' down disks
Z' disks under repair