Exam in EDA122(Chalmers) and DIT062(GU), Tuesday, August 16, 2011, 14:00 - 18:00

Problem 1

1.a

State Labelling

$\left\{\begin{array}{l} \textbf{A:}\text{Both "Active CM" and "Backup CM" are working} \\ \textbf{B:}\text{"Active CM" is faulty and "Backup CM" is active and working} \\ \textbf{C:}\text{"Backup CM" is faulty and "Active CM" is working} \\ \textbf{CF:}\text{Catastrophic failure state} \\ \textbf{SF:}\text{Safe failure state} \end{array}\right.$
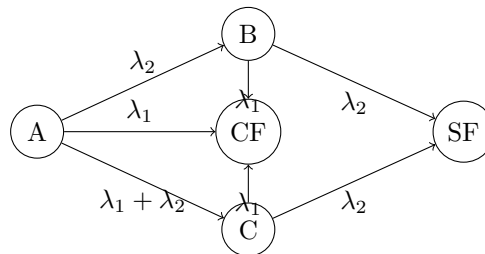
Figure 1: Markov chain

State B and state C can be merged together since they are having the same outgoing failure rates to states CF and SF. Figure 2, see next page, shows the simplified Markov Chain for this system.

State Labelling

$\left\{\begin{array}{l} \textbf{A:}\text{Both "Active CM" and "Backup CM" are working} \\ \textbf{B':}\text{Being in state B or state C of Markov model in Figure 1} \\ \textbf{CF:}\text{Catastrophic failure state} \\ \textbf{SF:}\text{Safe failure state} \end{array}\right.$

1.b

The steady-state safety of the system is

$$S(\infty) \quad = \quad \frac{2\lambda_2 + \lambda_1}{2\lambda_2 + 2\lambda_1} * \frac{\lambda_2}{\lambda_1 + \lambda_2}$$

1.c

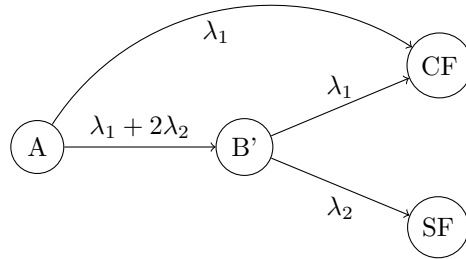We drive expressions for the probabilities being in state A and B'.

Figure 2: Simplified Markov chain

$$P'(t) = P(t)Q$$
$$P(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$
$$Q = \begin{bmatrix} -2\lambda_1 - 2\lambda_2 & 2\lambda_2 + \lambda_1 & 0 & \lambda_1 \\ 0 & -\lambda_1 - \lambda_2 & \lambda_2 & \lambda_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Laplace transform:

$$\mathcal{L}\{P'(t) = P(t)Q\} \Rightarrow sP(s) - P(0) = P(s)Q$$

$$\begin{cases} s\tilde{P}_A(s) - 1 = -(2\lambda_1 + 2\lambda_2)\tilde{P}_A(s) \\ s\tilde{P}_{B'}(s) = 2\lambda_2 + \lambda_1\tilde{P}_A(s) - (\lambda_1 + \lambda_2)\tilde{P}_B(s) \end{cases}$$

$$\tilde{P}_A(s) = \frac{1}{s + 2\lambda_1 + 2\lambda_2}$$
$$\tilde{P}_{B'}(s) = \frac{\lambda_1}{s + \lambda_1 + \lambda_2} * \frac{2\lambda_2 + \lambda_1}{s + 2\lambda_1 + 2\lambda_2}$$
$$\tilde{P}_{B'}(s) = \frac{X}{s + 2\lambda_1 + 2\lambda_2} * \frac{Y}{s + \lambda_1 + \lambda_2}$$

Inverse Laplace transform (see, e.g., Mathematics Handbook, pp. 332, L22.):

$$\mathcal{L}^{-1}\left\{\frac{n!}{(s+a)^{n+1}}\right\} = t^n e^{-at}$$

$$P_A(t) = \mathcal{L}^{-1}\left\{\frac{1}{s + 2\lambda_1 + 2\lambda_2}\right\} = e^{-(2\lambda_1 + 2\lambda_2)t}$$
$$P_{B'}(t) = \mathcal{L}^{-1}\left\{\frac{2\lambda_2 + \lambda_1}{(\lambda_1 + \lambda_2)}\left(\frac{1}{s + \lambda_1 + \lambda_2} - \frac{1}{s + 2\lambda_1 + 2\lambda_2}\right)\right\}$$

$$
\begin{aligned}
P_{B'}(t) &= \frac{2\lambda_2 + \lambda_1}{(\lambda_1 + \lambda_2)}(e^{-(\lambda_1+\lambda_2)t} - e^{-(2\lambda_1+2\lambda_2)t}) \\
\Rightarrow R(t) &= P_A(t) + P_{B'}(t) \\
R(t) &= e^{-2(\lambda_1+\lambda_2)*t} + (\frac{2\lambda_2 + \lambda_1}{\lambda_1 + \lambda_2}) * e^{-(\lambda_1+\lambda_2)*t} - e^{-(2\lambda_1+2\lambda_2)*t}
\end{aligned}
$$

Problem 2

Processors and disks can be considered as two independent subsystems of the system, so that their availabilities can be calculated separately. The availability of the whole system is the production of the availability of processors and disks.

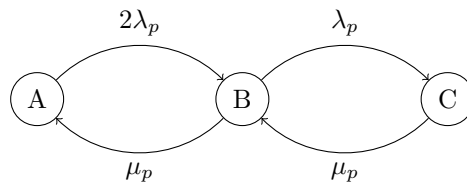The processors are modelled using the Markov-model shown in Figure 3.

Figure 3: Markov chain for Processors

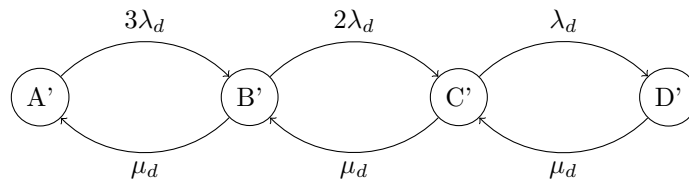The disks are modelled using the Markov-model shown in Figure 4.

Figure 4: Markov chain for Disks

This is a birth and death process (see, e.g., Mathematics Handbook, pp. 440-441). We can use the following formulas to calculate the steady state probability for state k.
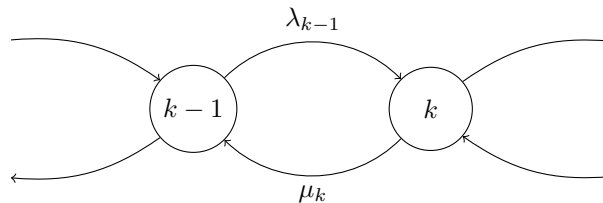

Figure 5: General birth and death process

$$\Pi_k = \frac{\lambda_{k-1}}{\mu_k}\Pi_{k-1}$$

$$\sum_k \Pi_k = 1$$

Using the formulas for a birth and death process, we get(for processors):

$$\begin{cases} \Pi_B & = & \frac{2\lambda_p}{\mu_p}\Pi_A \\ \Pi_C & = & \frac{\lambda_p}{\mu_p}\Pi_B \\ \Pi_A + \Pi_B + \Pi_C & = & 1 \end{cases}$$

$$\Pi_A = \frac{\mu_p^2}{\mu_p^2 + 2\lambda_p\mu_p + 2\lambda_p^2}$$

$$\Pi_B = \frac{2\lambda_p\mu_p}{\mu_p^2 + 2\lambda_p\mu_p + 2\lambda_p^2}$$

$$A_p(t) = \Pi_A + \Pi_B$$

$$A_p(t) = \frac{\mu_p^2 + 2\lambda_p\mu_p}{\mu_p^2 + 2\lambda_p\mu_p + 2\lambda_p^2}$$

Using the formulas for a birth and death process, we get(for disks):

$$\begin{cases} \Pi_{B'} & = & \frac{3\lambda_d}{\mu_d}\Pi_{A'} \\ \Pi_{C'} & = & \frac{2\lambda_d}{\mu_d}\Pi_{B'} = \frac{6\lambda_d^2}{\mu_d^2}\Pi_{A'} \\ \Pi_{D'} & = & \frac{\lambda_d}{\mu_d}\Pi_{C'} = \frac{6\lambda_d^3}{\mu_d^3}\Pi_{A'} \\ \Pi_{A'} + \Pi_{B'} + \Pi_{C'} + \Pi_{D'} & = & 1 \end{cases}$$

$$\Pi_{A'}\left(1 + \frac{3\lambda_d}{\mu_d} + \frac{2\lambda_d}{\mu_d}\frac{3\lambda_d}{\mu_d} + \frac{\lambda_d}{\mu_d}\frac{2\lambda_d}{\mu_d}\frac{3\lambda_d}{\mu_d}\right) = 1$$

$$\Longleftrightarrow$$

$$\Pi_{A'}\left(\frac{\mu_d^3 + 3\lambda_d\mu_d^2 + 6\lambda_d^2\mu_d + 6\lambda_d^3}{\mu_d^3}\right) = 1$$

$$A_d(t) = P_{A'(t)} + P_{B'(t)} + P_{C'(t)}$$

$$\lim_{t\to\infty} A_d(t) = \Pi_{A'(t)} + \Pi_{B'(t)} + \Pi_{C'(t)}$$

$$= \frac{\mu_d^3 + 3\lambda_d\mu_d^2 + 6\lambda_d^2\mu_d}{\mu_d^3 + 3\lambda_d\mu_d^2 + 6\lambda_d^2\mu_d + 6\lambda_d^3}$$

$$\Pi_{System} = \Pi_{Processors} * \Pi_{Disks}$$

Problem 3

Here the assumption is that the system components are repaired regardless of the type of failure (safe or unsafe failure) of the system. The system is described

by the GSPN model shown in Figure 6.

We define λ_3 as:

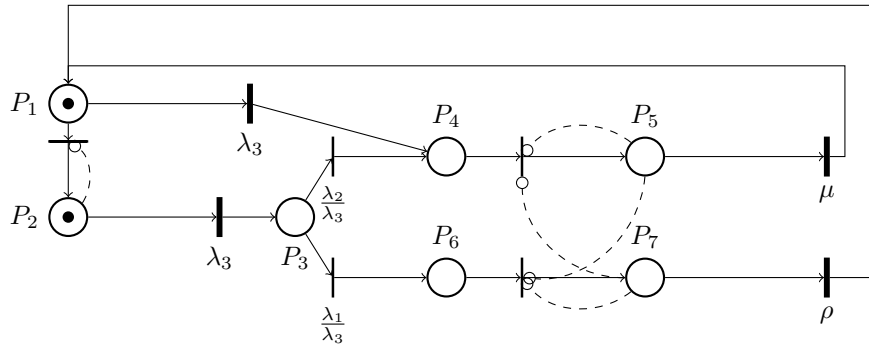$$\lambda_3 \quad = \quad \lambda_1 + \lambda_2$$

Figure 6: Petri Net for Problem 3

In Table 1 we define 10 states representing possible markings of the GSPN model. We draw the extended reachability graph according to the states defined in Table 1 .
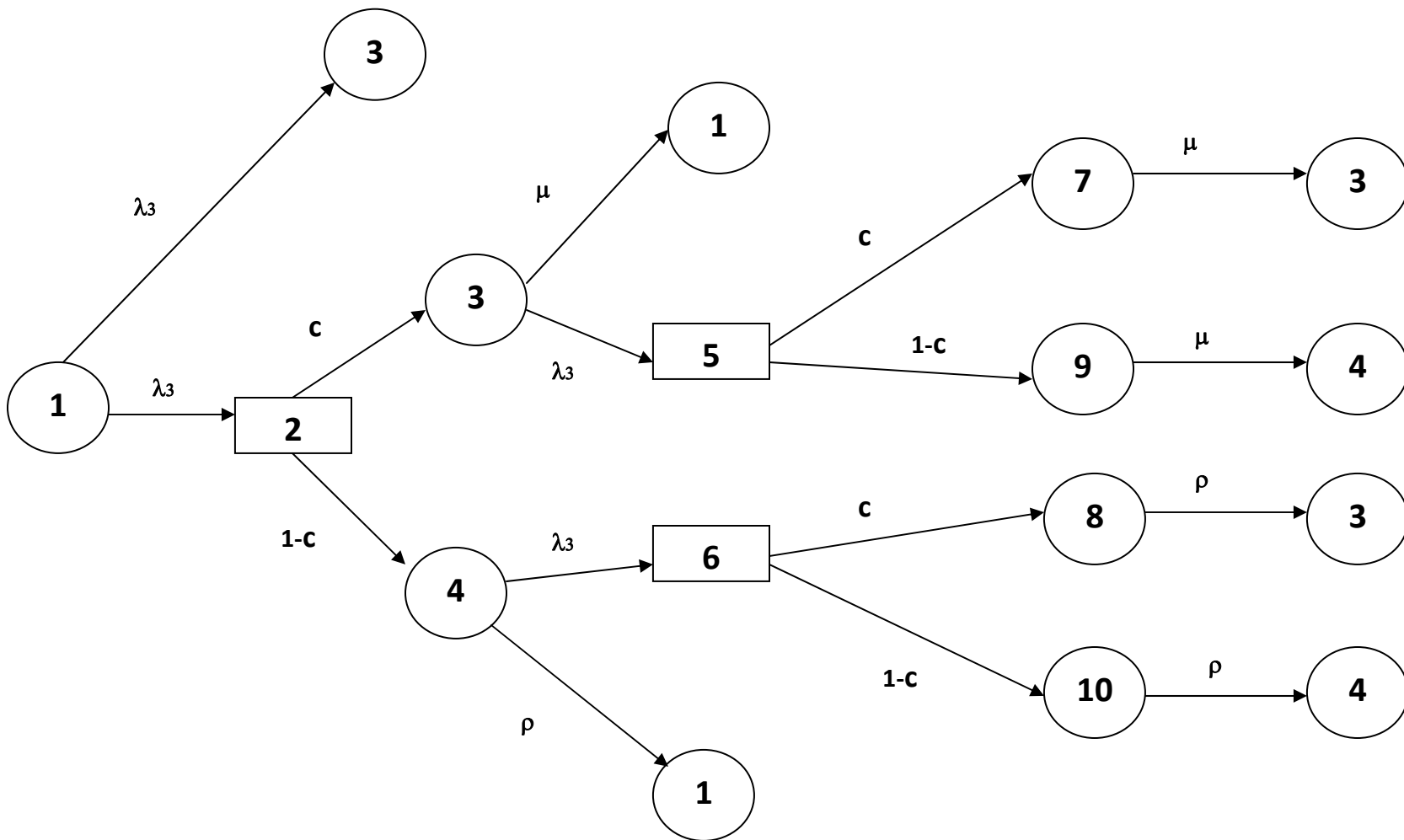
Table 1: Driving States for Reachability Graph

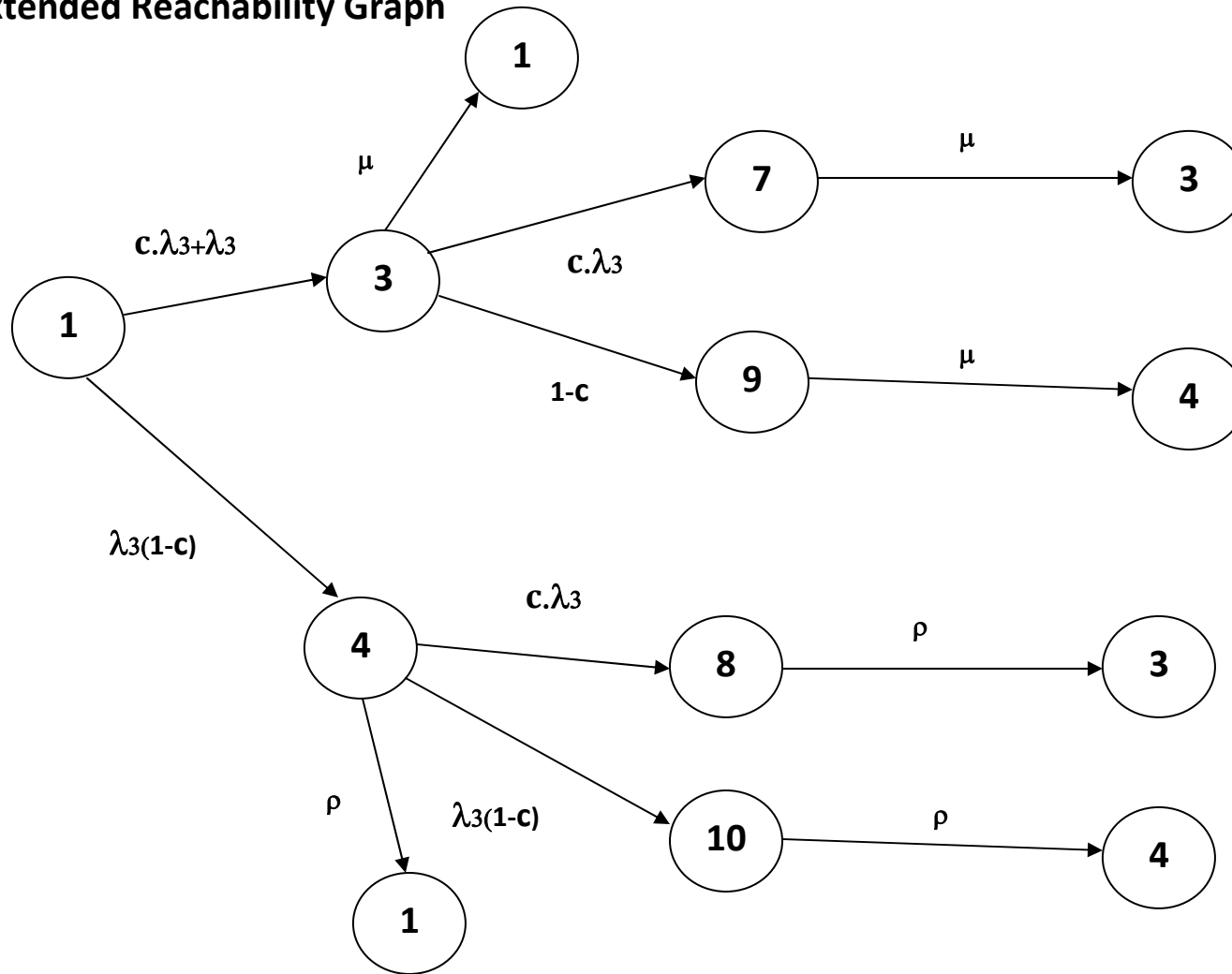| state	P_1	P_2	P_3	P_4	P_5	P_6	P_7
1	1	1	0	0	0	0	0
2	0	1	1	0	0	0	0
3	0	1	0	0	1	0	0
4	0	1	0	0	0	0	1
5	0	0	1	0	1	0	0
6	0	0	1	0	0	0	1
7	0	0	0	1	1	0	0
8	0	0	0	1	0	0	1
9	0	0	0	0	1	1	0
10	0	0	0	0	0	1	1

We define λ_3 as: $\lambda_3 = \lambda_1 + \lambda_2$

$C = \lambda_2 / \lambda_1 + \lambda_2$

Extended Reachability Graph

$1 \xrightarrow{\lambda_3} 2$

$1 \xrightarrow{\lambda_3} 3$

$2 \xrightarrow{c} 3$

$2 \xrightarrow{1-c} 4$

$3 \xrightarrow{\mu} 1$

$3 \xrightarrow{\lambda_3} 5$

$4 \xrightarrow{\lambda_3} 6$

$4 \xrightarrow{\rho} 1$

$5 \xrightarrow{c} 7$

$5 \xrightarrow{1-c} 9$

$6 \xrightarrow{c} 8$

$6 \xrightarrow{1-c} 10$

$7 \xrightarrow{\mu} 3$

$9 \xrightarrow{\mu} 4$

$8 \xrightarrow{\rho} 3$

$10 \xrightarrow{\rho} 4$

Extended Reachability Graph

Reachability Graph