Examination, Computability (DAT415/DIT310/DIT311/DIT312/TDA184)

- Date and time: 2020-01-15, 8:30–12:30.
- Author/examiner: Nils Anders Danielsson. Telephone number: 1680. Visits to the examination rooms: ∼9:30 and ∼11:30.
- Authorised aids (except for aids that are always permitted): None.
- The GU grades Pass (G) and Pass with Distinction (VG) correspond to the Chalmers grades 3 and 5, respectively.
- To get grade n on the exam you have to be awarded grade n or higher on at least n exercises.
- A completely correct solution of one exercise is awarded the grade 5. Solutions with minor mistakes *might* get the grade 5, and solutions with larger mistakes might get lower grades.
- Exercises can contain parts and/or requirements that are only required for a certain grade. To get grade n on such an exercise you have to get grade n or higher on every part marked with grade n or lower (and every unmarked part), and you have to fulfil every requirement marked with grade n or lower (as well as every unmarked requirement).
- Do not hand in solutions for several exercises on the same sheet.
- Write your examination code on each sheet.
- Solutions can be rejected if they are hard to read, unstructured, or poorly motivated.
- After correction the graded exams are available in the student office in room 4482 of the EDIT building. If you want to discuss the grading, contact the examiner no later than three weeks after the result has been reported. In this case you should not remove the exam from the student office.

1. (a) *For grade 3:* Give examples of sets A and B for which $A \times B$ is countable, whereas $A \to B$ is not. You do not need to provide proofs.

   (b) *For grade 4:* Either prove that the set

   $$\mathit{List}\ (\mathbb{N} \to \mathbb{N})$$

   is countable, or that it is not countable. Here $\mathit{List}\ A$ is the set containing *finite* lists of elements from the set A, as defined in the lectures. You can use theorems from the lecture slides without providing proofs for them.

2. Give concrete syntax for the χ expression e for which the standard χ encoding (as presented in the lectures), given using concrete syntax, is

   $$\begin{aligned}\ulcorner e \urcorner = \mathsf{Const}(&\mathsf{Zero}(),\\ &\mathsf{Cons}(\mathsf{Lambda}(\mathsf{Suc}(\mathsf{Zero}()),\\ &\qquad\mathsf{Const}(\mathsf{Zero}(),\\ &\qquad\quad\mathsf{Cons}(\mathsf{Var}(\mathsf{Suc}(\mathsf{Zero}())), \mathsf{Nil}()))),\\ &\quad\mathsf{Nil}()))).\end{aligned}$$

   Assume that the number 0 corresponds to the constructor True, and that the number 1 corresponds to the variable x.

3. If $f \in \mathbb{N} \rightharpoonup \mathbb{N}$ and $g \in \mathbb{N} \rightharpoonup \mathbb{N}$ are both χ-computable, is the partial function $h \in \mathbb{N} \rightharpoonup \mathbb{N}$ defined by $h\ n = f\ n + g\ n$ always χ-computable?

   For grade 3: Motivate your answer.

   For grade 4: Provide a proof. You are allowed to make use of Rice's theorem, the fact that the halting problem is undecidable, the fact that the *terminates-in* function from the lectures (which decides whether an expression terminates in at most a certain number of steps) is decidable, and the fact that addition of natural numbers is computable, but not other results stating that some function is or is not computable (unless you provide proofs).

   For grade 5: You may not use Rice's theorem (unless you provide a proof).

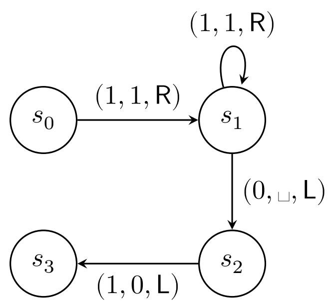
4. Is the following function χ-decidable?

   $$\begin{aligned}&f \in \mathit{CExp} \times \mathit{CExp} \to \mathit{Bool}\\ &f\,(e_1, e_2) = \textbf{if}\ \exists\, b \in \mathit{Bool}.\, [\![\mathsf{apply}\ e_1\ \ulcorner b \urcorner]\!] = [\![\mathsf{apply}\ e_2\ \ulcorner b \urcorner]\!]\\ &\qquad\qquad\ \ \textbf{then}\ \mathsf{true}\ \textbf{else}\ \mathsf{false}\end{aligned}$$

   The grade criteria of the previous exercise apply to this one as well.

5. Consider the following Turing machine:

- Input alphabet: $\{ 0, 1 \}$.
- Tape alphabet: $\{ 0, 1, \sqcup \}$.
- States: $\{ s_0, s_1, s_2, s_3 \}$.
- Initial state: s_0.
- Transition function:



(a) *For grade 3:* What is the result of running this Turing machine with 1110 as the input string? Does it halt? In that case, what is the resulting string?

(b) *For grade 4:* Let us represent natural numbers (0, 1, 2...) in the following way: the number $n \in \mathbb{N}$ is represented by a string with n ones followed by one zero (1^n0). Does this Turing machine witness the Turing-computability of some total function from $\mathbb{N}$ to $\mathbb{N}$? In either case you should provide a proof. If the answer is yes, then you should additionally give a simple description of the function that is witnessed, without any reference to Turing machines (no proof is needed for this part).

6. Consider the language RF$^-$ that we get if we remove suc from the abstract syntax of RF, as well as the rule of the semantics that refers to suc. Either prove that the semantics of RF$^-$ is total (what this means is defined more precisely below), or that it is not total.

   Here is the abstract syntax of RF$^-$:

$$\frac{}{\mathsf{zero} \in RF^-_0} \qquad \frac{i \in \mathbb{N} \qquad 0 \le i < n}{\mathsf{proj}\ i \in RF^-_n} \qquad \frac{f \in RF^-_m \qquad gs \in (RF^-_n)^m}{\mathsf{comp}\ f\ gs \in RF^-_n}$$

$$\frac{f \in RF^-_n \qquad g \in RF^-_{2+n}}{\mathsf{rec}\ f\ g \in RF^-_{1+n}} \qquad \frac{f \in RF^-_{1+n}}{\mathsf{min}\ f \in RF^-_n}$$

   The operational semantics of RF$^-$, $f[\rho] \Downarrow m$, is, for every $n \in \mathbb{N}$, a relation between programs $f \in RF^-_n$, vectors $\rho \in \mathbb{N}^n$, and natural numbers $m \in \mathbb{N}$, and $fs\,[\rho] \Downarrow^\star ms$ is, for all m, $n \in \mathbb{N}$, a relation between vectors $fs \in (RF^-_m)^n$, vectors $\rho \in \mathbb{N}^m$, and vectors $ms \in \mathbb{N}^n$. These two relations are defined inductively in the following way:

$$\frac{}{\mathsf{zero}\,[\mathsf{nil}] \Downarrow 0} \qquad \frac{}{\mathsf{proj}\ i\,[\rho] \Downarrow \mathit{index}\ \rho\ i} \qquad \frac{gs\,[\rho] \Downarrow^\star \rho' \qquad f[\rho'] \Downarrow n}{\mathsf{comp}\ f\ gs\,[\rho] \Downarrow n}$$

$$\frac{f[\rho] \Downarrow n}{\mathsf{rec}\ f\ g\,[\rho, \mathsf{zero}] \Downarrow n} \qquad \frac{\mathsf{rec}\ f\ g\,[\rho, m] \Downarrow n \qquad g\,[\rho, n, m] \Downarrow o}{\mathsf{rec}\ f\ g\,[\rho, \mathsf{suc}\ m] \Downarrow o}$$

$$\frac{f[\rho, n] \Downarrow 0 \qquad \forall m < n.\ \exists\, k \in \mathbb{N}.\, f[\rho, m] \Downarrow 1 + k}{\mathsf{min}\ f[\rho] \Downarrow n}$$

$$\frac{}{\mathsf{nil}\,[\rho] \Downarrow^\star \mathsf{nil}} \qquad \frac{fs\,[\rho] \Downarrow^\star ns \qquad f[\rho] \Downarrow n}{fs, f[\rho] \Downarrow^\star ns, n}$$

   The *index* function is defined in the following way (for any set A and natural number n):

$$\begin{aligned}
&\mathit{index} \in A^n \to \{\, i \in \mathbb{N} \mid 0 \le i < n \} \to A \\
&\mathit{index}\ (xs, x)\ \mathsf{zero} \qquad = x \\
&\mathit{index}\ (xs, x)\ (\mathsf{suc}\ i) = \mathit{index}\ xs\ i
\end{aligned}$$

   The semantics is total if, for every $n \in \mathbb{N}$, $f \in RF^-_n$ and $\rho \in \mathbb{N}^n$, there is some $m \in \mathbb{N}$ such that $f[\rho] \Downarrow m$.