# Domain Specific Languages of Mathematics
# Course codes: DAT326 / DIT982

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| **Results** | Announced within 19 days |
| **Exam check** | 2021-09-09, 12.15-12.45 |
| **Aids** | "all examination aids are allowed" (special covid-19 rules) |
| **Grades** | To pass you need **a minimum of 5p on each question (1 to 4)** and also reach these grade limits: 3: >=48p, 4: >=65p, 5: >=83p, max: 100p |

Remember to write legibly. Good luck!

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For reference: the learning outcomes. Some are tested by the hand-ins, some by the written exam.

- Knowledge and understanding
  - design and implement a DSL (Domain Specific Language) for a new domain
  - organize areas of mathematics in DSL terms
  - explain main concepts of elementary real and complex analysis, algebra, and linear algebra
- Skills and abilities
  - develop adequate notation for mathematical concepts
  - perform calculational proofs
  - use power series for solving differential equations
  - use Laplace transforms for solving differential equations
- Judgement and approach
  - discuss and compare different software implementations of mathematical concepts

1. [25p] **Typing maths: continuity and differentiability from limits**

   Consider continuity defined in terms of limits [Adams and Essex(2010), page 78]:

   > We say that a function $f$ is **continuous** at an interior point $c$ of its domain if
   > $\lim_{x \to c} (f\ x) = f\ c$
   > If either $\lim_{x \to c} (f\ x)$ fails to exist or it exists but is not equal to $f\ c$, then we will say that $f$ is **discontinuous** at $c$.

   and differentiability defined in terms of limits [Adams and Essex(2010), page 99].

   > The derivative of a function $f$ is another function $f'$ defined by
   > $f'\ x = \lim_{h \to 0} \frac{f(x+h) - f\ x}{h}$
   > at all points $x$ for which the limit exists (i.e., is a finite real number). If $f'\ x$ exists, we say that $f$ is **differentiable** at $x$.

   As an example, consider the function $s : \mathbb{R} \to \mathbb{R}$ where $s\ x = -x^2$ for negative $x$ and $s\ x = x^2$ otherwise.

   (a) [7p] Explain the typing and scoping rules for the *lim* construction (what can be written before and after the arrow, etc.).

   (b) [6p] Give the types of $f$, $f'$, $x$, $h$ in the definition of $f'\ x$. Explain your reasoning.

   (c) [6p] Prove that $s$ is differentiable at $x$ for all $x : \mathbb{R}$ using the definition above, and define a closed form for $s'$.

   (d) [6p] What is the set of all $x$ such that $s'$ is differentiable at $x$ and what is $s''$ (the 2nd derivative of $s$)?

2. [25p] **Proofs: Differentiability from continuity**

   Definition of differentiability adapted from [Pickert(1969)]:

   > Let $X \subseteq \mathbb{R}$, $a \in X$ and $f : X \to \mathbb{R}$. If there exists a function $\phi_f : X \to X \to \mathbb{R}$ such that, for all $x \in X$
   > $f\ x = f\ a + (x - a) * \phi_f\ a\ x$
   > such that $\phi_f\ a : X \to \mathbb{R}$ continuous at $a$, then $f$ is **differentiable** at $a$. The value $\phi_f\ a\ a$ is called the **derivative** of $f$ at $a$ and is denoted $f'\ a$.

   Note that for $X \subseteq \mathbb{R}$ we can define $\phi_f$ for $x \not\equiv a$ as follows:

   $\phi_f\ a\ x = (f\ x - f\ a)\ /\ (x - a)$

   but the definition above can also be generalised to work for vectors and matrices (when division is not available).

   As an example of this generalisation, let $V2 = \mathbb{R}^2$, $p_1, p_2 : V2 \to \mathbb{R}$, defined as $p_1\ (x_1, x_2) = x_1$ and $p_2\ (x_1, x_2) = x_2$. (Here $(*) : V2 \to V2 \to \mathbb{R}$ is the scalar product.)

   (a) [10p] Prove that $p_1$ is differentiable at $a$ for all $a : V2$ through these steps: What is the type of $\phi_{p_1}$? What is the definition of $\phi_{p_1}$ which satisfies the requirements? What is the derivative of $p_1$ at $(x_1, x_2)$?

   (b) [15p] (Return to the version where $X \subseteq \mathbb{R}$.) Prove that if $f$ and $g$ are both differentiable at $a$, then the product of $f$ and $g$ is also differentiable at $a$. In other words, use $\phi_f$ and $\phi_g$ to calculate a definition for $\phi_h$ satisfying the equation. Check that the derivative $h'\ a = \phi_h\ a\ a$ comes out as expected.

   Motivate your steps and make sure to keep track of types, scope, etc. of your expressions.

3. [25p] **Algebraic structure: abelian monoid**

   An **abelian monoid** is a set $M$ together with a constant (nullary operation) $0 \in M$ and a binary operation $\oplus : M \to M \to M$ (pronounced "oplus") such that:

   - 0 is a unit of $\oplus$

     $$\forall\, x \in M.\;\; x \oplus 0 = x \textit{ and } 0 \oplus x = x$$

   - $\oplus$ is associative

     $$\forall\, x, y, z \in M.\;\; x \oplus (y \oplus z) = (x \oplus y) \oplus z$$

   - $\oplus$ is commutative

     $$\forall\, x, y \in M.\;\; x \oplus y = y \oplus x$$

   (a) Define a type class *AbMonoid* that corresponds to the abelian monoid structure.

   (b) Define a datatype *AbMonoidExp v* for the language of abelian monoid expressions (with variables of type $v$) and define an *AbMonoid* instance for it. (These are expressions formed from applying the monoid operations to the appropriate number of arguments, e.g., all the left hand sides and right hand sides of the above equations.)

   (c) Find one other instance of the *AbMonoid* class and give an example which is a monoid but **not** an *AbMonoid*.

   (d) Define a general evaluator for *AbMonoidExp v* expressions on the basis of an assignment function.

   (e) Specialise the evaluator to the *AbMonoid* instance defined at point 3c. Take three *AbMonoidExp String* expressions, give the appropriate assignments and compute the results of evaluating the three expressions.

   Each question carries 5pts.

4. [25p] **Laplace**

   Consider the following coupled differential equations:

   $$\begin{aligned} f' &= f + g, & f(0) &= 1 \\ g' &= exp + 4g - 2f, & g(0) &= 3 \end{aligned}$$

   (a) [10p] Solve the equations assuming that $f$ and $g$ can be expressed by power series *fs* and *gs*, that is, use *integ* and the differential equations to express the relation between *fs*, *fs*′, *gs*, and *gs*′. What are the first three coefficients of *fs*? Explain how you compute them.

   (b) [15p] Solve the equations using the Laplace transform. You should need this formula and the rules for linearity + derivative:

   $$\mathscr{L}\,(\lambda t.\, e^{\alpha * t})\; s = 1/(s - \alpha)$$

   Show that your solutions do indeed satisfy the four requirements.

# References


[Adams and Essex(2010)] R. A. Adams and C. Essex. *Calculus: a complete course*. Pearson Canada, 7th edition, 2010.

[Pickert(1969)] G. Pickert. *Einführung in die Differential-und Integralrechnung*. Klett, 1969.