Domain Specific Languages of Mathematics
Course codes: DAT326 / DIT982

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Results	Announced within 19 days
Exam check	2021-03-26, 12.15-13.00
Aids	"all examination aids are allowed" (special covid-19 rules)
Grades	To pass you need **a minimum of 5p on each question (1 to 4)** and also reach these grade limits: 3: >=48p, 4: >=65p, 5: >=83p, max: 100p

Remember to write legibly. Good luck!

For reference: the learning outcomes. Some are tested by the hand-ins, some by the written exam.

- Knowledge and understanding
  - design and implement a DSL (Domain Specific Language) for a new domain
  - organize areas of mathematics in DSL terms
  - explain main concepts of elementary real and complex analysis, algebra, and linear algebra
- Skills and abilities
  - develop adequate notation for mathematical concepts
  - perform calculational proofs
  - use power series for solving differential equations
  - use Laplace transforms for solving differential equations
- Judgement and approach
  - discuss and compare different software implementations of mathematical concepts

1. [25p] **Algebraic structure:** *ring* (lightly edited from the Wikipedia entry)

   > ...a *ring* is a set R equipped with two binary operations satisfying properties analogous to those of addition and multiplication of integers. Ring elements may be numbers such as integers or complex numbers, but they may also be non-numerical objects such as polynomials, square matrices, functions, and power series.
   >
   > Formally, a ring is an abelian group whose operation is called addition, with a second binary operation called multiplication that is associative, is distributive over the addition operation, and has a multiplicative identity element.

   Some of the laws are (for all a, b, c in R):

   $$
   \begin{aligned}
   &(a + b) + c = a + (b + c) \\
   &a + 0 = a \\
   &a + (-a) = 0 \\
   &a \cdot (b + c) = (a \cdot b) + (a \cdot c)
   \end{aligned}
   $$

   (a) Define a type class *Ring* that corresponds to the structure *ring*.

   (b) Define a datatype $R\ v$ for the language of ring expressions (with variables of type v) and define a *Ring* instance for it. (These are expressions formed from applying the ring operations to the appropriate number of arguments, e.g., all the left hand sides and right hand sides of the above equations.)

   (c) Find and implement two other instances of the *Ring* class. Make sure the laws are satisfied.

   (d) Give a type signature for, and define, a general evaluator for $R\ v$ expressions on the basis of an assignment function.

   (e) Specialise the evaluator to the two *Ring* instances defined in (1c). Take three ring expressions (of type *Ring String*), give the appropriate assignments and compute the results of evaluating, in each case, the three expressions.

   Each question carries 5pts.

2. [25p] **Laplace**

   Consider the following differential equation:

   $$2f + 2f' + f'' = 0, \quad f(0) = 1, \quad f'(0) = -1$$

   (a) [10p] Solve the equation assuming that f can be expressed by a power series *fs*, that is, use *integ* and the differential equation to express the relation between *fs*, *fs*′, *fs*″. What are the first four coefficients of *fs*? Explain how you compute them.

   (b) [15p] Solve the equation using the Laplace transform. You should need this formula (note that α can be a complex number) and the rules for linearity + derivative:

   $$\mathscr{L}\,(\lambda t.\, e^{\alpha * t})\ s = 1/(s - \alpha)$$

   Show that your solution does indeed satisfy the three requirements.

3. [25p] **Proofs and homomorphisms:**

   That a function $h : A \to B$ is a homomorphism from $Op : A \to A \to A$ to $op : B \to B \to B$ can be abbreviated as:

   $$H_2\ (h, Op, op) = \forall\ x.\ \ \forall\ y.\ \ h\ (Op\ x\ y) == op\ (h\ x)\ (h\ y)$$

   (a) [5p] Prove $\exists\ op.\ \ H_2\ (odd, (+), op)$ where $odd :: \mathbb{Z} \to Bool$ checks if a number is odd.

   (b) [10p] Prove or disprove $\exists\ add.\ \ H_2\ (degree, (+), add)$ where $degree :: Poly\ \mathbb{R} \to Maybe\ \mathbb{N}$ computes *Just* the degree of a polynomial (or *Nothing* for the zero polyomial).

   (c) [10p] With $D : Fun \to Fun$ being the usual derivative operator for functions of one argument ($Fun = \mathbb{R} \to \mathbb{R}$), prove $\neg\ (\exists\ mul.\ \ H_2\ (D, (*), mul))$, where $(*)$ is pointwise multplication of functions.

   Motivate your steps and make sure to keep track of types, scope, etc. of your expressions.

4. [25p] **Typing maths: differentials**

   Consider the following (slightly edited) quote from [Adams, p105]:

   > The Newton quotient $[f(x+h)-f(x)]/h$, whose limit we take to find the derivative dy/dx, can be written in the form $\Delta y/\Delta x$, where [...].
   > The Newton quotient $\Delta y/\Delta x$ is actually the quotient of two quantities, Δy and Δx. It is not at all clear, however, that the derivative dy/dx, the limit of $\Delta y/\Delta x$ as Δx approaches zero, can be regarded as a quotient. If y is a continuous function of x, then Δy approaches zero when Δx approaches zero, so dy/dx appears to be the meaningless quantity 0/0. Nevertheless, it is sometimes useful to be able to refer to quantities dy and dx in such a way that quotient is the derivative dy/dx. We can justify this by regarding dx as a new *independent* variable (called **the differential of** x) and defining a new *dependent* variable dy (**the differential of** y) as a function of x and dx by
   >
   > $$dy = \frac{dy}{dx}dx = f'(x)dx.$$

   (a) [10p] Give the types of x, y, f, dy/dx, dx, dy, f'. Explain your reasoning.

   (b) [5p] What is dy if $y = f(x^2)$?

   (c) [10p] What would be the corresponding notion of differential dz for a two-variable function $z = g(x_1, x_2)$? It should be a function of $2 * 2$ independent variables.