Domain Specific Languages of Mathematics
Course codes: DAT326 / DIT982

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Results	Announced within 19 days
Exam check	by email appointment (patrikj@chalmers.se)
Aids	"all examination aids are allowed" (special covid-19 rules)
Grades	To pass you need **a minimum of 5p on each question (1 to 4)** and also reach these grade limits: 3: >=48p, 4: >=65p, 5: >=83p, max: 100p

Remember to write legibly. Good luck!

For reference: the learning outcomes. Some are tested by the hand-ins, some by the written exam.

- Knowledge and understanding
  - design and implement a DSL (Domain Specific Language) for a new domain
  - organize areas of mathematics in DSL terms
  - explain main concepts of elementary real and complex analysis, algebra, and linear algebra
- Skills and abilities
  - develop adequate notation for mathematical concepts
  - perform calculational proofs
  - use power series for solving differential equations
  - use Laplace transforms for solving differential equations
- Judgement and approach
  - discuss and compare different software implementations of mathematical concepts

1. [30p] **Algebraic structure:** a DSL for vector spaces

   A vector space over $\mathbb{R}$ is a set V together with a constant (or nullary) operation $0 : V$ (called "zero"), an operation $(+) : V \to V \to V$ (called "add"), and an *external* operation $(\cdot) : \mathbb{R} \to V \to V$ (called "scale"), such that

   - 0 is the unit of $(+)$:
     $\forall\, v \in V.\; v + 0 = 0 + v = v$
   - $(+)$ is associative:
     $\forall\, v_1, v_2, v_3 \in V.\; (v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$
   - $(+)$ is invertible:
     $\forall\, v \in V.\; \exists\, (-v) \in V.\; v + (-v) = (-v) + v = 0$
   - $(+)$ is commutative:
     $\forall\, v_1, v_2 \in V.\; v_1 + v_2 = v_2 + v_1$
   - $(\cdot)$ is "associative"
     $\forall\, x_1, x_2 \in \mathbb{R}, v \in V.\; x_1 \cdot (x_2 \cdot v) = (x_1 * x_2) \cdot v$
     Remark: $(*)$ denotes the standard multiplication in $\mathbb{R}$
   - 1 is a unit of $(\cdot)$:
     $\forall\, v \in V.\; 1 \cdot v = v$
   - $(\cdot)$ distributes over $(+)$:
     $\forall\, x \in \mathbb{R}, v_1, v_2 \in V.\; x \cdot (v_1 + v_2) = x \cdot v_1 + x \cdot v_2$
   - $(\cdot)$ distributes over $(+)$
     $\forall\, x_1, x_2 \in \mathbb{R}, v \in V.\; (x_1 + x_2) \cdot v = x_1 \cdot v + x_2 \cdot v$

   (a) Define a type class *Vector* that corresponds to the structure "vector space over $\mathbb{R}$".

   (b) Define a datatype *VecSyn a* for the language of vector space expressions (with variables of type *a*) and define a *Vector* instance for it. (These are expressions formed from applying the quasigroup operations to the appropriate number of arguments, e.g., all the left hand sides and right hand sides of the above equations.)

   (c) Find and implement two other instances of the *Vector* class. Make sure the laws are satisfied.

   (d) Give a type signature for, and define, a general evaluator for *VecSyn a* expressions on the basis of an assignment function.

   (e) Specialise the evaluator to the two *Vector* instances defined in (1c). Take three vector expressions (of type *VecSyn String*), give the appropriate assignments and compute the results of evaluating, in each case, the three expressions.

   Each question carries 6pts.

2. [20p] **Typing maths:** Derivative of Inverse

   Consider the following (slightly edited) quote from [Adams, p167]:

   > **Derivatives of Inverse Functions**
   > Suppose that the function f is differentiable on an interval (a, b) and that either $f'(x) > 0$ for $a < x < b$, so that f is increasing on (a, b), or $f'(x) < 0$ for $a < x < b$, so that f is decreasing on (a, b). In either case f is one-to-one on (a, b) and has an inverse, f^{-1} , defined by
   >
   > $$y = f^{-1}(x) \iff x = f(y), \quad (a < y < b).$$
   >
   > [... example graph skipped ...]
   > Therefore, [... calculation skipped ...] and
   >
   > $$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

   (a) [5p] Give types for f, a, b, x, y, f^{-1}, and f'. Make sure to account for the possibility that the types of x and y could be different.

   (b) [7p] Give (short, textual) names and types to d/dx and to the function $\cdot^{-1}$ that takes a function to its inverse.

   (c) [8p] Restate the final equation twice using your new names, first as a direct translation, then in point-free style (with no mention of x). You may use $\mathit{recip}\ x = 1\ /\ x$.

3. [25p] **Inverses and proofs**

   Let $n = d + 2$ where d is the last digit of your personal identity number. (For example, if 19890102-3286 would sit the exam, d would be 6 and $n = 8$.)

   (a) [13p] Consider the function $f(x) = n + x^n$ on the interval $(0, 1)$. Check that the conditions from the quote in problem 2 are applicable to your f. Compute f' and f^{-1} and state their domains and ranges. Show that the last equation in problem 2 holds for your f.

   (b) [12p] Prove the equation for a general f (satisfying the conditions).

4. [25p] **Laplace**

   Consider the following differential equation:

   $$f''(t) + \sqrt{3} * f'(t) = 6 * (f(t) - 1), \quad f0 = 4, \quad f'0 = 0$$

   (a) [10p] Solve the equation assuming that f can be expressed by a power series fs, that is, use integ and the differential equation to express the relation between fs, fs', fs'', and the power series rhs for the right hand side. What are the first four coefficients of fs?

   (b) [15p] Solve the equation using the Laplace transform. You should need this formula (note that α can be zero) and the rules for linearity + derivative:

   $$\mathscr{L}\,(\lambda t.\, e^{\alpha * t})\; s = 1/(s - \alpha)$$

   Show that your solution does indeed satisfy the three requirements.