Domain Specific Languages of Mathematics
Course codes: DAT326 / DIT982

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Results Announced within 19 days

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Aids One textbook of your choice (e.g., Adams and Essex, or Rudin, or Beta - Mathematics Handbook). No printouts, no lecture notes, no notebooks, etc.

Grades 3: 40p, 4: 60p, 5: 80p, max: 100p

Remember to write legibly. Good luck!

For reference: the DSLsofMath learning outcomes. Some are tested by the hand-ins, some by the written exam.

- Knowledge and understanding
  - design and implement a DSL (Domain Specific Language) for a new domain
  - organize areas of mathematics in DSL terms
  - explain main concepts of elementary real and complex analysis, algebra, and linear algebra
- Skills and abilities
  - develop adequate notation for mathematical concepts
  - perform calculational proofs
  - use power series for solving differential equations
  - use Laplace transforms for solving differential equations
- Judgement and approach
  - discuss and compare different software implementations of mathematical concepts

1. [20pts] Algebraic structure: a DSL for monoids

   - $(R, +, 0)$ is a monoid with identity element 0:
     $(a + b) + c = a + (b + c)$
     $0 + a = a + 0 = a$

   (a) Define a type class *Monoid* that corresponds to the monoid structure.

   (b) Define a datatype *ME v* for the language of monoid expressions (with variables of type v) and define a *Monoid* instance for it. (These are expressions formed from applying the monoid operations to the appropriate number of arguments, e.g., all the left hand sides and right hand sides of the above equations.)

   (c) Find two other instances of the *Monoid* class.

   (d) Give a type signature for, and define, a general evaluator for *ME v* expressions on the basis of an assignment function.

   (e) Specialise the evaluator to the two *Monoid* instances defined in (1c). Take three monoid expressions of type *ME String*, give the appropriate assignments and compute the results of evaluating, in each case, the three expressions.

   Each question carries 4pts.

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2. [30pts] Consider the following quote from "A Companion to Analysis" by Thomas Körner:

   > Definition 1.43. Let U be an open set in $\mathbb{R}$. We say that a function $f : U \to \mathbb{R}$ is differentiable at $t \in U$ with derivative $f'\ (t)$ if, given $\epsilon > 0$, we can find a $\delta\ (t, \epsilon) > 0$ such that $(t - \delta\ (t, \epsilon), t + \delta\ (t, \epsilon)) \subseteq U$ and
   > $|\frac{f\ (t+h) - f\ (t)}{h} - f'\ (t)| < \epsilon$
   > whenever $0 < |h| < \delta\ (t, \epsilon)$.

   (a) Give types for the symbols U, f, t, f', ϵ, δ.

   (b) Define the corresponding logical predicate *DifferentiableAt* (f, f', t). Explain briefly where the variables are bound.

   (c) Define a function δ_2 that shows *DifferentiableAt* (sq, tw, t) where $sq\ x = x * x$ and $tw\ x = 2 * x$.

   (d) Consider $P \Rightarrow Q$ with $P =$ *DifferentiableAt* (f, f', t) & *DifferentiableAt* (g, g', t) and $Q =$ *DifferentiableAt* $(f + g, f' + g', t)$. Sketch how δ_{f+g} (needed in Q) can be defined in terms of δ_f and δ_g (used in P).

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3. [20pts] Consider the following differential equation:

$$f(t) = \frac{f''(t) + f'(t)}{2}, \quad f(0) = a, \quad f'(0) = b$$

(a) [10pts] Solve the equation assuming that f can be expressed by a power series fs, that is, use *integ* and the differential equation to express the relation between fs, fs', and fs''. What are the first three coefficients of fs (expressed in terms of a and b)?

(b) [15pts] Solve the equation using the Laplace transform. You should need this formula (and the rules for linearity + derivative):

$$\mathscr{L}\,(\lambda t.\, e^{\alpha * t})\; s = 1/(s - \alpha)$$

Show that your solution does indeed satisfy the three requirements.

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4. [30pts] **Homomorphisms.** Consider the following predicates:

```
-- h : A → B is a homomorphism from E : A to e : B
H₀ (h, E, e)    = h E == e
-- h : A → B is a homomorphism from F : A → A to f : B → B
H₁ (h, F, f)    = ∀ x.  h (F x) == f (h x)
-- h : A → B is a homomorphism from Op : A → A → A to op : B → B → B
H₂ (h, Op, op) = ∀ x.  ∀ y.  h (Op x y) == op (h x) (h y)
```

And the following definitions of $\mathbb{R}$ and $\mathbb{C}$ as vector spaces:

```
class Vector v where
  zero   :: v
  add    :: v → v → v
  scale  :: ℝ → (v → v)
type ℂ = ComplexSem ℝ
instance Vector ℝ where zero = 0;       add = (+);   scale = (*)
instance Vector ℂ where zero = zeroC; add = addC; scale = scaleC
newtype ComplexSem r = CS (r, r) deriving (Eq, Show)
i = CS (0, 1)
zeroC = CS (0, 0)
addC (CS (x₁, y₁)) (CS (x₂, y₂)) = CS (x₁ + x₂, y₁ + y₂)
scaleC r (CS (x, y)) = CS (r * x, r * y)
toComplex :: ℝ → ComplexSem ℝ
toComplex r = CS (r, 0)
mulC (CS (x₁, y₁)) (CS (x₂, y₂)) = CS (x₁ * x₂ − y₁ * y₂, x₁ * y₂ + x₂ * y₁)
circle :: ℝ → ℂ
circle v = CS (cos v, sin v)
```

Prove or disprove the following claims:

(a) H_0 (*toComplex*, *zero*, *zero*)

(b) H_2 (*toComplex*, *add*, *add*)

(c) H_2 (*toComplex*, *scale*, *scale*)

(d) H_2 (*circle*, $(+)$, *mulC*). You may use trigonometric identities.

(e) $\exists\; a, b.\;\; H_1$ (*addC i*, *mulC a*, *mulC b*)