Solutions to Jan 12 exam
Mixed-Signal Systems (DAT116)

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Jan 12, 2018

The following are suggested solutions to the problems. It must be noted that other solutions may be possible, and that approximate solutions will give at least partial credit as long as the approximations are reasonable (just as in real life!).

1. *A single-sinewave signal of varying amplitude and a frequency less than 10 MHz is to be sampled and converted to digital form. The peak-SNR requirement is 72 dB.*

   (a) *Suggest minimum values of sample frequency and quantizer resolution to fulfill the specification.*

   The sample frequency must fulfill the Nyquist criterion and thus be at least 2×10^7 Hz (in practice, a larger value will be preferable to relax the pre-sampling filter requirements).

   As we are dealing with a *single* sine wave, the peak SNR (which occurs at full-range amplitude) will be given by $6.02 \cdot N + 1.76$ dB. $N = 11$ would yield only 67.98 dB, so we need to go to 12 bits (which would result in a peak SNR of 74 dB).

   (b) *The same setup, with parameters as per task 1a, is used to convert a signal $x(t)$ composed of two equal-amplitude sine waves:*

   $$x(t) = A(sin(2\pi f_1 t) + sin(2\pi f_2 t))$$

   f_1 and f_2 are both less than 10 MHz but are **not** *rationally related. Estimate the resulting peak SNR.*

   Since the frequencies of two sine waves are *not* rationally related, the peak value of the sum will be equal to the sum of the peak values of the constituent sine waves (if the peaks never coincide, there must be a rational frequency relation!); so $A_{max} = 0.5$. Different-frequency sine waves are uncorrelated, so the total signal power is given by the sum of the powers of the two waves:

   $$P_{signal} = P_{f1} + P_{f2} = \frac{(\frac{A}{2})^2}{2} + \frac{(\frac{A}{2})^2}{2} = \frac{A^2}{4}$$

   That is, half the power of a single sine wave with the same peak value. With the quantization noise at $\Delta^2/12$ in both cases, the SNR will be 3 dB worse than in the single-tone case, at $6.02 \times 12 + 1.76 - 3 = 71$ dB.

(c) *Suggest and motivate sample-clock jitter requirements for the two conversion situations above.*

In the single-tone case, for the jitter error to have the same power as the quantization error, the timing inaccuracy would be given by

$$\Delta t_{rms} = \frac{10^{-\frac{74}{20}}}{(2\pi \cdot 10^7)}$$

so $\Delta t_{rms} = 3.17$ ps.

In the two-tone case, since each of the tones might have a frequency arbitrarily close to 10^7, the maximum jitter-induced error will be the same in the two-tone case, as will the Δt_{rms} requirement (but since the signal power is lower, the SNDR will be lower, as discussed in task 1b).

[The derivations above will give full marks. A more careful derivation could include combining two unequal error powers, as has been discussed in class.]

2. *Emilia needs to A/D-convert a baseband signal with a bandwidth of 500 kHz. An unused 4-bit quantizer is available in the system she works on; but she needs to apply noise shaping to improve the SNDR.*

(a) *Aware that noise-shaping feedback loops may misbehave at high input power levels, Emilia decides to limit her input signal to 3 dB below full scale. What peak SNDR should she expect for a first-order loop filter, a sample rate of 16 MHz, and appropriate digital filters to remove the out-of-band noise?*

The linear-model formula suggests that the peak SNDR should be $6.02 \times 4 + 1.76 - 5.17 + 9.03 \times \log_2(16) = 56.79$ dB (in practice, it may be somewhat lower). Emilia's input power reduction will reduce this figure by a further 3 dB, to at most 53.79 dB.

(b) *To suppress spurious tones at low DC inputs, Emilia decides to try adding some white dither noise to the input signal. What noise power might she apply without destroying the SNDR at higher input power levels? Note: you need to specify the noise bandwidth.*

Since the noise is added *to the input signal*, is will not be suppressed by the noise shaping. The noise that makes it through to the output must be less than the shaped and filtered quantization noise, so more than 56.79 dB below the power of a full-scale sinewave (potentially reduced further to be insignificant next to the quantization noise). This is the power *within the band of interest*; if the bandwidth of the noise extends to 16× the signal bandwidth (due to the 16× oversampling), the total noise power could be $\log_2(16) \times 3 = 4 \times 3 = 12$ dB higher.

(c) *Emilia fears that her feedback DAC might suffer from a third-order nonlinearity:*

$$y = x - \alpha x^3$$

If the full range of the input signal is ±1, what is the maximum acceptable value of α? Motivate.

Errors in the feedback DAC will also not be suppressed by noise shaping. If $x = A \sin \omega t$, then

$$\begin{aligned} y &= A \sin \omega t - \alpha A^3 \sin^3 \omega t \\ &= A \sin \omega t - \alpha A^3 \left(\frac{3 \sin \omega t - \sin 3\omega t}{4} \right) \\ &= (A - \frac{3\alpha}{4} A^3) \sin \omega t + \frac{\alpha}{4} A^3 \sin 3\omega t \end{aligned}$$

Clearly, the third harmonic was added by the nonlinearity. Harmonics rise faster than the fundamental with increasing input power, so the worst case should be at the highest input power, that is, at $A = 1/\sqrt(2) = 0.71$ (due to Emilia's 3-dB reduction). The third tone has then been reduced by 9 dB due to the third power of the amplitude. Thus, the limit on α is given by

$$\alpha < 4 \times 10^{-\frac{56.79-(9-3)}{20}} \approx 0.011$$

3. *The figure below (from Pelgrom: Analog-to-Digital conversion, Springer 2010) shows a very simple D-to-A converter, and timing diagrams for the operation of the switches.* ...

   (a) *Derive the output voltage as a function of the input bits under the assumption that $C_1 = C_2$.*
   With each new added bit value, the existing V_{out} is halved and the new bit value is added to it. Thus the end result after N cycles will be

   $$V_{out}(N) = V_{REF+} \cdot \sum_{i=1}^{N} \frac{b_i}{2^{N-i}}$$

   (b) *Identical capacitors can be closely matched (at an area cost). Estimate the ENOB value possible with a converter like this if C_1 and C_2 can be matched to within 0.1%.*
   This is clearly a binary-scaling converter. A relative error of C_1 vs C_2 will cause a corresponding error in the value of the current bit compared to all the lesser bits. We recognize the characteristics of the R2R converter! 0.1% is one part in 1000, or roughly 1 part in 2^{10}, so we will not be able to go beyond an ENOB of 10.

   (c) *The converter is operated at a rate of 10^6 complete conversions per second, with a resolution corresponding to the ENOB you calculated in the previous task (please assume an ENOB value if you did not solve the previous task). Assume that V_{REF+} is 2 V. How large must each capacitor be for the kT/C noise to be insignificant when compared with the matching error?*
   With $V_{REF+} = 2$ V, the range of output values stretches from 0 to 2 V. A full-scale sinewave signal can therefore at most have an amplitude of 1, which means a power of 0.5.
   The power of the quantization error will be $10 \times 6 + 2 = 62$ dB below this value, so a factor of $10^{-62/10} = 6.3 \times 10^{-7}$ below the full scale, or $P_Q = 3.2 \times 10^{-7}$. A value of $C = kT/P_Q = 4.14 \times 10^{-21}/3.2 \times 10^{-7} \approx 13$ fF will make the powers comparable; a clearly larger capacitance than this is needed to make the kT/C noise insignificant. (The capacitances may also need to be larger for the matching to be as good as indicated in the previous task; bigger capacitances improve both matching and noise!)

[Solution corrected since last use of this problem]

4. *Medical implants such as heart pacemakers have stringent low-power requirements, both to maximize battery life and to minimize heating of surrounding tissues. A recent PhD thesis describes an 8-bit, 11 kS/s ADC with a power dissipation of $2.83\mu W$. The INL is quoted as ≈ 1 LSB, and the SNDR as 47 dB for a 1-kHz signal 0.2 dB below full scale.*

   (a) *The quoted power dissipation is very low; but speed and resolution requirements are not very challenging. How does this design compare to the state-of-the-art in low-power converters?*
   The sample-related power of a quantizer at this performance level is limited by $P_S = 12kTf_S2^{2N}$. With $k = 1.38 \times 10^{-23}$ J/K, $T = 300$ K, $f_S = 11 \times 10^3$ Hz, and $N = 8$, $P_S = 35$ pW. The quoted dissipation is therefore more than 3.66×10^5 larger than the sample power, whereas the most frugal converters published get by with factors less than 1000. Clearly the designer had concerns even more important than further reducing the already-low dissipation.

(b) *With the simplifying assumption that the INL is due to a second-order nonlinearity, calculate the input-related IP2 for the converter.*

Assume that the square-function shape of the INL curve for the second-order nonlinearity is symmetric around the midpoint of the range, such that

$$y = x + \alpha x^2$$

(cf. task 2c above!). Then a maximum INL value of 1 LSB means that the converted value at each end of the range deviates by 1 LSB from the nominal value. For an 8-b converter, this means that $\alpha = 2^{-7}$, and that a full-scale sine wave input $x = \sin \omega t$ (with amplitude 1) will generate an output $y = \sin \omega t + 2^{-8}(1 - \cos 2\omega t)$. The amplitude of the second-harmonic signal is seen to be 8 factors of 2 below that of the fundamental, corresponding to 48 dB; that also means that IP2 is 48 dB *above* the full-range input.

(c) *What limits the peak SNDR of the converter: the quantization error, the nonlinearity, or something else?*

According to the standard formula, the quantization error should be $6.02 \times 8 + 1.76 \approx 50$ dB below the full-scale single-sinewave signal. This observation poses a problem: two uncorrelated errors at –50 dB would result in the given overall SNDR of –47 dB; but the nonlinearity seems to contribute with –48 dB on its own! It seems one of our assumptions must be incorrect:

- The INL was maybe not caused by a second-order nonlinearity
- Nonlinearity and quantization error were maybe not uncorrelated
- There may have been measurement or roundoff errors in some of the reported dB values

Certainly there is no need to postulate a jitter error to explain the overall SNDR (and with a modest resolution and very low signal frequencies, jitter problems are highly unlikely).