Solutions to Jan 16 exam
Mixed-Signal Systems (DAT116)

Lars Svensson

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The following are suggested solutions to the problems. It must be noted that other solutions may be possible, and that approximate solutions will give at least partial credit as long as the approximations are reasonable (just as in real life!).

1. This type of converter is known as a *algorithmic* converter. It is conceptually simple to extend to high resolutions, but several real-life effects limit it to medium-performance applications, as will be seen below.

   (a) Assuming no undersampling, a signal frequency of 10 MHz means a sample rate of at least 20 MHz. A clock frequency of 300 MHz then means that 15 bit decisions can be made before the next sample arrives. A 15-bit resolution corresponds to an SNDR of $6.02 \times 15 + 1.76 = 92$ dB for a full-scale sinewave.

   (b) We can easily calculate the sample jitter that would give an error equal to the quantization error above:

   $$92 = -20\log(2\pi \cdot 10^7 \cdot \Delta t_{RMS})$$

   $$\Delta t_{RMS} = 4 \times 10^{-13}$$

   That is, the RMS value of the sample-time error must be less than half a picosecond. Additionally, if these errors were combined, we could expect an overall SNDR of $92 - 3 = 89$ dB.

   (c) In a binary-scaling architecture such as this one, the accuracy of the first multiplication by two is critical: a one-percent error here will be larger than one unit in the seventh bit (since $1/2^7 = 1/128 < 0.01$). Likewise, 15-bit precision would require an accuracy of no worse than $1/2^{15} = 1/32768 \approx 3 \times 10^{-5}$. It would be very difficult to reach such precision of passive components without laser trimming or active dynamic trimming.

   (d) We need to know the size of the sampling capacitance. We don't know the matching properties of the process, so knowledge about matching requirements does not help us to determine the capacitance. However, the sample capacitance is in any case limited by the kT/C noise:

   $$C_S > \frac{12kT}{V_{FS}^2} \cdot 2^{2N} = \frac{1}{V_{FS}^2} \cdot 52pF$$

   To be able to drive this capacitance across the full-scale voltage each clock cycle, the required power is:

   $$P = V_{FS}^2 \cdot C_S \cdot f_{ck} = 52 \times 10^{-12} \cdot 3 \times 10^8 = 156 \times 10^{-4} = 15\text{mW}$$

The total dissipation requirement for the entire converter is likely to be at least 200 times larger, so might be in the neighborhood of 3W.

2. The $\Sigma\Delta$ DAC may be viewed as a variation of the ADC, with the analog voltages replaced with high-precision digital values. The resolution of the final DAC then corresponds to the quantizer resolution in the ADC case.

   (a) The system bandwidth is 15 MHz; thus, the Nyquist frequency is 30 MHz. The final DAC can perform 900M conversions per second, so the maximum OSR is $900/30 = 30$. Based on linear loop models, we know (Maloberti, Eq. 6.16) that

   $$SNR_{PEAK,1} = 6.02 \cdot N + 1.76 - 5.17 + 9.03 \cdot log2(OSR_1) = 6.02 \cdot N + 40.89$$

   The SNDR is 53 dB at 2 dB below the full-scale output, which would correspond to 55 dB at full scale. Thus, the minimum resolution in bits[1] would be given by

   $$N = \left\lceil \frac{55 - 40.89}{6.02} \right\rceil = 3$$

   (b) The second-order-loop peak SNR is given in Maloberti Eq. 6.25 as

   $$SNR_{PEAK,2} = 6.02 \cdot N + 1.76 - 12.9 + 15.05 \cdot log2(OSR_2)$$

   Equalizing the two SNR expressions and solving for OSR_2 yields

   $$OSR_2 = 11$$

   Thus, it appears that the clock rate might be reduced by a factor of $30/11 = 2.72$.

   (c) A 3-bit converter may be built from $2^3 - 1 = 7$ sources; the binary value $j, 0 \leq j < 8$ would then cause the "first" j sources to be switched in:

   $$i_{tot} = \sum_{k=1}^{j} i_k$$

   In this case, the currents are not all equal to the nominal unit current i_{nom}; rather, the current values depend on k. Let ξ be the ratio of the first and the last of these; then:

   $$\begin{aligned} i_{tot} &= \sum_{k=1}^{j} i_{nom} \cdot (1 + k \cdot \frac{\xi}{7}) = && (1) \\ &= i_{nom}(j + \frac{\xi}{7} \sum_{k=1}^{j} k) = && (2) \\ &= i_{nom}(j + \frac{\xi}{7} \cdot \frac{(j+1)j}{2}) = && (3) \\ &= i_{nom}(j + \frac{\xi}{7}(\frac{j^2}{2} + \frac{j}{2})) \approx && (4) \\ &\approx i_{nom}(j + j^2 \frac{\xi}{14}) && (5) \end{aligned}$$

   where the second (square) term is the error.

   Clearly i_{tot} is not a linear function of j, and the error has a square characteristic. If the gain error is disregarded, the largest deviation from the ideal straight line occurs at the middle of the range, and is one fourth of the full-scale error (at $j = 7$), that is

   $$\frac{1}{4} i_{nom} \frac{49}{14} \xi = i_{nom} \frac{49}{56} \xi$$

[1] In task 2c further down, it is revealed that the DAC is implemented with a thermometer-coded row of sources; thus the number of levels is really not restricted to a power of two. But "3 bits" will be good for full marks here.

which must be the peak-to-peak value (twice the amplitude) of the error (at twice the frequency of an input sine) for a full-scale input signal. Thus, the error *power* at full scale would be

$$\left(\frac{1}{2} i_{nom} \frac{49}{56} \xi\right)^2 /2 \approx 0.095 i_{nom}^2 \xi^2$$

The full-scale *signal* power is given by the full-range amplitude:

$$\frac{3.5^2 i_{nom}^2}{2} = 6.125 i_{nom}^2$$

Finally, the SNDR is given by:

$$\frac{6.125 i_{nom}^2}{0.095 i_{nom}^2 \xi^2} \approx 64/\xi^2$$

The target SNDR value of 55 dB corresponds to a power ratio of 3.16×10^5, which corresponds to $\xi = 0.014$. Thus, the ratio between the first and the last current values must be no larger than 1.4%.

3. (a) `R2` and `R3` determine the DC gain through the expression $A_{DC} = 1 + R_3/R_2$. Unless Q is small, it may be approximated as the ratio of the peak gain A_{peak} and the DC gain of the filter section. Thus,

$$Q = \frac{A_{peak}}{A_{DC}}$$

and also

$$Q = \frac{1}{3 - A_{DC}}$$

so

$$A_{peak} = Q \cdot A_{DC} = \frac{A_{DC}}{3 - A_{DC}}$$

and

$$A_{DC} = \frac{3A_{peak}}{1 + A_{peak}}$$

The nominal magnitude function of the filter just touches the upper passband gain limit at the peak of the magnitude function of the highest-Q filter section. This is the point that would be most affected by a Q-value change. A 0.5-dB change would mean a magnitude change by a factor of $10^{0.5/20} = 1.06$, i.e., by 6%. The nominal value for A_{DC} would give $A_{peak} = 12.6$; a 6% increase for A_{peak} would give $A_{peak} = 13.39$ and $A_{DC} = 2.79$, so $R_3/R_2 = 1.79$. Clearly, the matching requirements for R_3 and R_4 are very strict, at 1 part in 179, or 0.56%.

(b) The limited amplifier gain causes a reduction in overall gain due the discrepancy, given by

$$D = \frac{A\beta}{1 + A\beta}$$

As raw gain falls off at higher frequencies, the worst case is at the passband edge, at 5kHz. Due to the limited GBW, at that frequency, the raw gain of the op-amp is roughly $3000/5 = 600$.

At the passband edge, the filter-section gain will be 2dB below the peak (a factor of 80%), so $0.8 \cdot 12.6 = 10.0$. With $A = 600$ and $\beta = 0.1$, $D = 0.983$, so the gain can be expected to be 1.7% lower than what β set it to, even with perfect resistor values. A gain increase as described in the previous task would cause violation at the peak gain before the passband edge would be cleared.

4. (a) The two sinewaves have different frequencies, so the maximum and minimum values of the two sinewaves will sometimes be in phase and sometimes out of phase. In the former case, their combined amplitudes will have twice the amplitude of each signal on its own, so each signal has one-half of the full-scale-sinewave amplitude. Thus, their powers are one fourth of the full-scale-sinewave power.

   (b) As we seek a ratio between the intended signal and its distortion products, and as all terms in the expression for $y(t)$ contain G as a factor, we may without limitation assume that $G = 1$. We then have

   $$y(t) = (x(t) - \alpha \cdot (x(t))^3$$

   Since $x(t) = \frac{1}{2}\sin\omega_1 t + \frac{1}{2}\sin\omega_2 t$, we have

   $$(x(t))^3 = \frac{1}{2^3}\left(\sin^3\omega_1 t + 3\sin^2\omega_1 t\sin\omega_2 t + 3\sin\omega_1 t\sin^2\omega_2 t + \sin^3\omega_2 t\right)$$

   Since $\sin 3\phi = 3\sin\phi - 4\sin^3\phi$, the first and last terms will cause out-of-band frequency components which we ignore, plus a contribution at ω_1 and ω_2 respectively; since α is small, we ignore also the latter next to the linear term in the first equation.

   The remaining terms in the second equation have the form $\sin^2\omega_a t \cdot \sin\omega_b t$. Since $\sin^2\omega_a t = (1 - \cos 2\omega_a t)/2$, each term causes a contribution at one of the signal frequencies (ignored as above), an out-of-band term (ignored as above), and a term with frequency $2\omega_a - \omega_b$, which is close to the inteded frequencies and cannot be filtered out. These are the sideband error terms to include in the SNDR. The amplitude of each of these two terms is

   $$\alpha \cdot \frac{1}{8} \cdot 3 \cdot \frac{1}{2} = \alpha\frac{3}{16}$$

   The power of each term is then

   $$\alpha^2\frac{9}{256}$$

   The SNDR is the power at the original frequencies (consisting of 2 equal parts) divided by the power of the sidebands (also 2 equal parts):

   $$SNDR = \frac{\frac{1}{4}\cdot\frac{1}{2}}{\alpha^2\frac{9}{256}} = \frac{1}{\alpha^2}\cdot\frac{64}{9} \approx \frac{7.11}{\alpha^2}$$

   As an example, with $\alpha = 0.01$, then $SNDR = \frac{7.11}{0.0001} = 7.11 \times 10^4$, which corresponds to about 48 dB.

   (c) The power of the quantization error should be no larger than the power of the sideband tones:

   $$2 \cdot \alpha^2\frac{9}{256} \geq \frac{\Delta^2}{12} = \frac{1}{12}\cdot\left(\frac{2}{2^N}\right)^2 = \frac{1}{3}\cdot 2^{-2N}$$

   which specifies the minimum resolution N as a function of α. As an example, $\alpha = 0.01$ gives a minimum N value of 8 (which seems to agree with the SNDR value in the previous task).