Logic in Computer Science

DAT060/DIT202 (7.5 hec)

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Monday 24th of October 2022, 8:30–12:30

Total: 60 points	
CTH: $\geqslant 30$: 3, $\geqslant 41$: 4, $\geqslant 51$: 5	GU: $\geqslant$30: G, $\geqslant$46: VG

No help material but dictionaries to/from English.

Write in English and as readable as possible (think that what we cannot read we cannot correct).

OBS: All answers should be ***carefully*** motivated. Points will be deduced when you give an unnecessarily complicated solution or when you do not properly justify your answer.

Good luck!

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1. Give proofs in natural deduction of the following sequents:

   (a) (3pts) $\neg s \to \neg r, (p \wedge q) \vee r, \neg s \to \neg q \vdash \neg p \vee s$

   Solution:

1.	$\neg s \to \neg r$	premise
2.	$(p \wedge q) \vee r$	premise
3.	$\neg s \to \neg q$	premise
4.	$p \wedge q$	assumption
5.	q	$\wedge e_2$ 4
6.	$\neg\neg q$	$\neg\neg$i 5
7.	$\neg\neg s$	MT (3,6)
8.	r	assumption
9.	$\neg\neg r$	$\neg\neg$i 8
10.	$\neg\neg s$	MT (3,9)
11.	$\neg\neg s$	$\vee$e (2,4–7,8–10)
12.	s	$\neg\neg$e 11
13.	$\neg p \vee s$	$\vee i_2$ 12

(b) (3.5pts) $\vdash \neg(p \wedge (\neg p \vee q)) \vee q$

Solution:

1.	$q \vee \neg q$	LEM
2.	q	assumption
3.	$\neg(p \wedge (\neg p \vee q)) \vee q$	$\vee i_2$ 2
4.	$\neg q$	assumption
5.	$p \wedge (\neg p \vee q)$	assumption
6.	$p \vee \neg p$	LEM
7.	p	assumption
8.	$\neg p \vee q$	$\wedge e_2$ 5
9.	$\neg p$	assumption
10.	$\bot$	$\neg e$ (9,7)
11.	q	assumption
12.	$\bot$	$\neg e$ (4,11)
13.	$\bot$	$\vee e$ (8,9–10,11–12))
14.	$\neg p$	assumption
15.	p	$\wedge e_1$ 5
16.	$\bot$	$\neg e$ (14,15)
17.	$\bot$	$\vee e$ (6,7–13,14–16)
18.	$\neg(p \wedge (\neg p \vee q))$	$\neg i$ (5–17)
19.	$\neg(p \wedge (\neg p \vee q)) \vee q$	$\vee i_1$ 18
20.	$\neg(p \wedge (\neg p \vee q)) \vee q$	$\vee e$ (1,2–3,4–19)

2. (a) (1pt) Without using truth tables, find the only valuation that makes the following formula satisfiable:

$$(\neg p \vee q) \wedge (p \vee r) \wedge (q \rightarrow \neg r \wedge \neg p)$$

(b) (2.5pts) Explain how you arrived to your solution and why no other valuation actually makes the formula satisfiable.

Solution:

(a) When p and q are false and r is true then the formula is true and hence the formula is satisfiable (that is, there is an interpretation that satisfies the formula).

(b) For the formula to be true, all three parts of the conjunction need to be true.

The rightmost part is an implication and there are 2 possible ways to make an implication true:

- both q and $\neg r \wedge \neg p$ are true, which means that both r and p need to be false since their negation both need to be true given that they are connected by a conjunction.
  But if r and p are false then $p \vee r$ is also false and then the whole formula becomes false as well.
  So this is not a possible valuation.
- q is false, which this makes the whole implication true.
  Since q is false then p must necessarily also be false so that $\neg p \vee q$ becomes true.
  Now, r necessarily needs to be true so that $p \vee r$ becomes true.

3. For each of the sequents below, prove using natural deduction that they are valid, or give a counter-model showing that they are not.

(a) (2pts) $\vdash \forall x.\exists y.x = y$

Solution:

1.	x_0	fresh
2.	$x_0 = x_0$	=i on x_0
3.	$\exists y.x_0 = y$	∃i 2
4.	$\forall x.\exists y.x = y$	∀i (1–3)

(b) (3pts) $\exists x.Q(x) \wedge \forall x.(P(x) \to \neg Q(x)) \vdash \exists x.\neg P(x)$

Solution:

1.	$\exists x.Q(x) \wedge \forall x.(P(x) \rightarrow \neg Q(x))$	premise
2.	$\exists x.Q(x)$	$\wedge e_1$ 1
3.	$\forall x.(P(x) \rightarrow \neg Q(x))$	$\wedge e_2$ 1
4.	x_0	fresh
5.	$Q(x_0)$	assumption
6.	$P(x_0) \rightarrow \neg Q(x_0)$	$\forall$e 3 with x_0
7.	$P(x_0)$	assumption
8.	$\neg Q(x_0)$	$\rightarrow$e (6,7)
9.	$\bot$	$\neg$e (8,5)
10.	$\neg P(x_0)$	$\neg$i (7–9)
11.	$\exists x.\neg P(x)$	$\exists$i 10
12.	$\exists x.\neg P(x)$	$\exists$e (2,4–11)

(c) (3pts) $\forall x.(P(x) \rightarrow \exists y.Q(x,y)) \vdash \exists x.(P(x) \rightarrow \forall y.Q(x,y))$

Solution:
The sequent is not valid. We will give a counter-model.
Consider $\mathcal{A} = \mathbb{N}$, $P^{\mathcal{M}}$ the set of even numbes, and $Q^{\mathcal{M}}$ the set of pairs (a,b) such that $a = 2b$.
In this model the premise holds since for any even natural i there is another natural number j such that $i = 2j$.
On the other hand, there is no even natural numbers k such that $k = 2s$ for all other natural numbers s.

(d) (3pts) $\forall x.(P(x,x) \vee \forall y.Q(x,y)) \vdash \forall x.(\exists y.P(x,y) \vee Q(x,x))$

Solution:

1.	$\forall x.(P(x,x) \vee \forall y.Q(x,y))$	premise
2.	x_0	fresh
3.	$P(x_0,x_0) \vee \forall y.Q(x_0,y))$	$\forall$e 1 with x_0
4.	$P(x_0,x_0)$	assumption
5.	$\exists y.P(x_0,y)$	$\exists$i 4
6.	$\exists y.P(x_0,y) \vee Q(x_0,x_0)$	$\vee$i$_1$ 5
7.	$\forall y.Q(x_0,y))$	assumption
8.	$Q(x_0,x_0)$	$\forall$e 7 with x_0
9.	$\exists y.P(x_0,y) \vee Q(x_0,x_0)$	$\vee$i$_2$ 8
10.	$\exists y.P(x_0,y) \vee Q(x_0,x_0)$	$\vee$e (3,4–6,7–9)
11.	$\forall x.(\exists y.P(x,y) \vee Q(x,x))$	$\forall$i (2–10)

4. Consider the following semantic entailments

   (i) $\forall x.(P(x) \to \neg Q(x)) \models \exists x.(P(x) \wedge \neg Q(x))$

   (ii) $\forall x.(P(x) \to \neg Q(x)) \models \neg\exists x.(P(x) \wedge Q(x))$

   (iii) $\forall x.(P(x) \to \neg Q(x)), \forall x.\forall y.(Q(x) \to R(x,y)) \models \forall x.\forall y.(P(x) \wedge Q(y) \to R(x,y))$

   (a) (1.5pt) What is a model for the language?

   Solution:
   A model $\mathcal{M}$ for the language consists in a domain $\mathcal{A} \neq \emptyset$, two unary relations $P^{\mathcal{M}}, Q^{\mathcal{M}} \subseteq \mathcal{A}$ and a binary relation $R^{\mathcal{M}} \subseteq \mathcal{A} \times \mathcal{A}$.

   (b) (3x3pts) Explain semantically (that is, reasoning with models) whether the above entailments hold or not.

   Solution:

   (i) This entailment is not valid.
   Consider a model $\mathcal{M}$ with domain $\mathcal{A} \neq \emptyset$, $P^{\mathcal{M}} = \emptyset$ and $Q^{\mathcal{M}} \subseteq \mathcal{A}$.
   In this model, the premise $\forall x.(P(x) \to \neg Q(x))$ is valid simply because no element in the domain satisfies the condition of the implication given $P^{\mathcal{M}} = \emptyset$. Since $P^{\mathcal{M}} = \emptyset$, there cannot be any element $a \in \mathcal{A}$ such that $a \in P^{\mathcal{M}}$, and hence $\exists x.P(x)$ does not hold in this model. Nor can $\exists x.(P(x) \wedge \neg Q(x))$ hold either.

(ii) This entailment is valid.
Consider a model $\mathcal{M}$ such that $\mathcal{M} \models \forall x.(P(x) \to \neg Q(x))$.
This tells us that for all $a \in \mathcal{A}$, whenever $a \in P^{\mathcal{M}}$ then $a \notin Q^{\mathcal{M}}$.
So, there cannot be an $a \in \mathcal{A}$ such that $a \in P^{\mathcal{M}}$ and also $a \in Q^{\mathcal{M}}$.
Hence, $\neg\exists x.(P(x) \wedge Q(x))$ holds in the model.

(iii) The entailment is not valid.
Consider a model $\mathcal{M}$ with $\mathcal{A} = \{1,2,3\}, P^{\mathcal{M}} = \{1,3\}, Q^{\mathcal{M}} = \{2\}$ and $R^{\mathcal{M}} = \{(2,1),(2,2),(2,3)\}$.
The premises hold in this model: for all $a \in \mathcal{A}$, if $a \in P^{\mathcal{M}}$ then $a \notin Q^{\mathcal{M}}$, and for all $a \in \mathcal{A}$, if $a \in Q^{\mathcal{M}}$ we have that $(a,b) \in R^{\mathcal{M}}$ for all $b \in \mathcal{A}$.
On the other hand, the conclusion doesn't hold in the model: we have that $1 \in P^{\mathcal{M}}$ and $2 \in Q\mathcal{M}$ but $(1,2) \notin R^{\mathcal{M}}$.

5. Consider the following theory, on a language with one constant a and one unary function symbol f. We write $f^2(x)$ for $f(f(x))$, $f^3(x)$ for $f(f(f(x)))$ and so on.

$$\psi_0 = R(a,a) \qquad \psi_1 = \forall x \forall y\ (R(x,y) \to R(f(x), f(f(y))))$$

(a) (1pt) Can we show $\psi_0,\ \psi_1 \vdash R(f(a), f^2(a))$?

(b) (3pts) Can we show $\psi_0,\ \psi_1 \vdash R(f(a), f(a))$?

(c) (2pts) When do we have $\psi_0,\ \psi_1 \vdash R(f^n(a), f^m(a))$?

Solution:

(a) From ψ_1 we get $R(a,a) \to R(f(a), f^2(a))$ and by modus ponens/$\to$e with ψ_0 we get $R(f(a), f^2(a))$.

(b) We give a counter example to show that we do not have $\psi_0,\ \psi_1 \vdash R(f(a), f(a))$.
Let the model be such that the domain is the set of natural numbers, $a^M = 0$, $f^M(x) = x+1$ and $R^M(x,y)$ means $y = 2x$. In this model the formula $R(f(a), f(a))$ does not hold.
Hence by soundness, this formula cannot be proved from ψ_0 and ψ_1.

(c) If we consider the model above, it follows that if $\psi_0,\ \psi_1 \vdash R(f^n(a), f^m(a))$, then $m = 2n$. Conversely, we can show $\psi_0,\ \psi_1 \vdash R(f^n(a), f^{2n}(a))$ by induction on n.

6. (2pts) Let A be a finite set and Φ a function $Pow(A) \to Pow(A)$, where $Pow(A)$ is the set of subsets of A. We assume that we have $\Phi(Y) \subseteq \Phi(X)$ whenever $X \subseteq Y$. Prove that there exists a subset X of A such that $\Phi(\Phi(X)) = X$.

Solution:
We have $\Psi(Y) = \Phi(\Phi(Y))$ monotone: let $X \subseteq Y$ then $\Phi(Y) \subseteq \Phi(X)$ and $\Phi(\Phi(X)) \subseteq \Phi(\Phi(Y))$, hence $\Psi(X) \subseteq \Psi(Y)$.

Hence by Tarski fixpoint Theorem there exists X such that $\Psi(X) = X$.

7. (a) (1.5pts) Explain what is a model for LTL.
   (b) (3pts) Explain when an LTL formula is *valid* for a given model with the example of why the formula $(FGa \wedge FGb) \rightarrow FG(a \wedge b)$ is valid for any model.
   (c) (3pts) Give a model where the formula $(GFa \wedge GFb) \rightarrow GF(a \wedge b)$ is *not* valid.

Solution:

(a) A model is a tuple $(S, \rightarrow, P, L)$ where $S, \rightarrow$ is a finite graph such that $N(s) \neq \emptyset$ for all s, P is a set of atomic formulae and $L : S \rightarrow Pow(P)$.

(b) A formula ψ is valid for such model if it is valid for all path on this model. If we have $\pi, 0 \models FGa \wedge FGb$ we have k such that a in $L(\pi(n))$ for $k \leqslant n$ and l such that b in $L(\pi(n))$ for $l \leqslant n$. For $max(k, l) \leqslant n$ we have both a and b in $L(\pi(n))$ and so $\pi, 0 \models FG(a \wedge b)$.

(c) If we take $S = \{s_0, s_1\}$ and $s_0 \rightarrow s_1$ and $s_1 \rightarrow s_0$ and $L(s_0) = \{a\}$ and $L(s_1) = \{b\}$ and $\pi = s_0 \rightarrow s_1 \rightarrow s_0 \rightarrow s_1 \dots$ we have $\pi, 0 \models GFa$ and $\pi, 0 \models GFb$ but not $\pi, 0 \models GF(a \wedge b)$.

8. Are the following LTL formulae valid
   (a) (2pts) $F(Xp \rightarrow p)$?
   (b) (2pts) $Fp \rightarrow (\neg p\ U\ p)$?

Solution:

(a) The formula is valid. If we have a model M and a path π, then $\pi, n \models Xp \rightarrow p$ is not valid only if $\pi, n+1 \models p$ and not $\pi, n \models p$. So we either have $\pi, 0 \models Xp \rightarrow p$ or $\pi, 1 \models Xp \rightarrow p$.

(b) The formula is valid. If we have $\pi, 0 \models Fp$ and we take the *first* n such that $\pi, n \models p$ we have not $\pi, k \models p$ for $k < n$ and $\pi, n \models p$ and hence $\pi, 0 \models (\neg p\ U\ p)$.

9. Are the following CTL formulae valid

   (a) (3pts) $(EGa \wedge EGb) \to EG(a \wedge b)$?

   (b) (3pts) $(AF(a \vee b) \wedge AG(a \to AFb)) \to AFb$?

   (c) (3pts) $(AF(a \vee b) \wedge EG(a \to EFb)) \to EFb$?

Solution:

(a) The formula is not valid. For a counter model take $S = \{s_0, s_1, s_2\}$ with $s_0 \to s_1$ and $s_0 \to s_2$ and $s_i \to s_i$ for $i = 1, 2$ and take $L(s_0) = \{a, b\}$ and $L(s_1) = \{a\}$ and $L(s_2) = \{b\}$. We have then $s_0 \models EGa$ and $s_1 \models EGb$ but we don't have $s_0 \models EG(a \wedge b)$.

(b) The formula is valid. Given $s \models AF(a \vee b)$ and $s \models AG(a \to AFb)$ and a path π starting from s we have n such that $\pi(n) \models a \vee b$ and $\pi(n) \models a \to AF(b)$. We then have $\pi(n) \models b$ or $\pi(n) \models a$ and then $\pi(n) \models AF(b)$. In all cases, we have $k \geqslant n$ such that $\pi(n + k) \models b$.

(c) The formula is valid. Given s such that $s \models AF(a \vee b)$ and $s \models EG(a \to EFb)$ we can find a path π starting from s such that $\pi(n) \models a \to EFb$ on this path. Since $s \models AF(a \vee b)$ we have $\pi(N) \models a \vee b$ for some N. If we have $\pi(N) \models b$ we have $s \models b$. If we have $\pi(N) \models a$ we have $\pi(N) \models EFb$ and hence $s \models EFb$.