Logic in Computer Science

DAT060/DIT202/DIT201 (7.5 hec)

Responsible: Ana Bove – Telefon: 1020

Monday 3rd of January 2021, 14:00–18:00

Total: 60 points	
CTH: $\geqslant$ 30: 3, $\geqslant$ 41: 4, $\geqslant$ 51: 5	GU: $\geqslant$30: G, $\geqslant$46: VG

No help material but dictionaries to/from English.

Write in English and as readable as possible (think that what we cannot read we cannot correct).

OBS: All answers should be ***carefully*** motivated. Points will be deduced when you give an unnecessarily complicated solution or when you do not properly justify your answer.

Good luck!

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1. (2x3pts) Give proofs in natural deduction of the following sequents:

   (a) $p \to (q \vee r) \vdash \neg q \to (p \to r)$

   Solution:

1.	$p \to (q \vee r)$	premise
2.	$\neg q$	assumption
3.	p	assumption
4.	$q \vee r$	$\to$e (1,3)
5.	q	assumption
6.	$\bot$	$\to$e (2,5)
7.	r	$\bot$e 6
8.	r	assumption
9.	r	$\vee$e (4,5–7,8–8)
10.	$p \to r$	$\to$i (3–9)
11.	$\neg q \to (p \to r)$	$\to$i (2–10)

   (b) $\vdash \neg(p \leftrightarrow \neg p)$

Solution:
Recall this is the same as $\vdash \neg(p \to \neg p \wedge \neg p \to p)$

1.	$p \to \neg p \wedge \neg p \to p$	assumption
2.	$p \to \neg p$	$\wedge e_1$ 1
3.	$\neg p \to p$	$\wedge e_2$ 1
4.	$p \vee \neg p$	LEM
5.	p	assumption
6.	$\neg p$	$\to$e (2,5)
7.	$\bot$	$\to$e (6,5)
8.	$\neg p$	assumption
9.	p	$\to$e (3,8)
10.	$\bot$	$\to$e (8,9)
11.	$\bot$	$\vee$e (4,5–7,8–10)
12.	$\neg(p \leftrightarrow \neg p)$	$\to$i 1–11

2. (a) (1pts) Without using truth tables, give a valuation where p and s have the same value but where NOT ALL the propositional atoms have the same value, for which the following formula is true:

$$(p \leftrightarrow r \wedge s) \wedge (q \vee s \to r) \wedge (p \vee q \to s)$$

(b) (2.5pts) Explain the reasoning and your answer both in the case where p and s are true and in the case they are false.

Solution:

(a) p, s and q false and r true, or p, s and r true, and q false.

(b) For the formula to be true all parts of the conjunction need to be true.

For $(p \leftrightarrow r \wedge s)$ to be true then p and $r \wedge s$ need to have the same value.

If p and s are false, then $r \wedge s$ is false independent of r.
Now, for the formula $(p \vee q \to s)$ to be true then q needs to be false as well.
Since q and s are false then $(q \vee s \to r)$ is true independent of r so we could set r to true.

If p is true then $r \wedge s$ needs to be true and hence both r and s need to be true. Fortunately, with these 3 proposiontal atoms true, no matter the value of q, the formulas $q \vee s \to r$ and $p \vee q \to s$ become true so we can set q false.

3. (3x3pts) For each of the sequents below, prove using natural deduction that they are valid, or give a counter-model showing that they are not.

(a) $\forall x.\forall y.R(x,y) \vdash \forall x.\forall y.(R(x,y) \land R(y,x))$

Solution:

1.	$\forall x.\forall y.R(x,y)$	premise
2.	x_0	fresh
3.	y_0	fresh
4.	$\forall y.R(x_0,y)$	$\forall$e 1 with x_0
5.	$R(x_0,y_0)$	$\forall$e 4 with y_0
6.	$\forall y.R(y_0,y)$	$\forall$e 1 with y_0
7.	$R(y_0,x_0)$	$\forall$e 6 with x_0
8.	$R(x_0,y_0) \land R(y_0,x_0)$	$\land$i (5,7)
9.	$\forall y.(R(x_0,y) \land R(y,x_0))$	$\forall$i (3–8)
10.	$\forall x.\forall y.(R(x,y) \land R(y,x))$	$\forall$i (2–9)

(b) $\forall x.\neg\forall y.R(x,y) \vdash \neg\forall x.\forall y.\neg R(x,y)$

Solution:
We will give a counter-model.
Consider $\mathcal{A} = \mathbb{N}$ and $R^{\mathcal{M}} \equiv \emptyset$. In this model the formula $\forall x.\neg\forall y.R(x,y)$ is valid since it is indeed the case that for any $n \in \mathbb{N}$, not for all $m \in \mathbb{N}$ we have $(n,m) \in R^{\mathcal{M}}$ simply because $R^{\mathcal{M}} \equiv \emptyset$.
The formula $\neg\forall x.\forall y.\neg R(x,y)$ estates that it is not the case that for all $n,m \in \mathbb{N}$, $(n,m) \notin R^{\mathcal{M}}$. This means that there must be at least a pair $(n,m) \in R^{\mathcal{M}}$.
But since $R^{\mathcal{M}}$ is empty there are actually no $n,m \in \mathbb{N}$ such $(n,m) \in R^{\mathcal{M}}$.
Observe that in any model where $R^{\mathcal{M}}$ is not empty, the formula $\neg\forall x.\forall y.\neg R(x,y)$ holds since it is logically equivalent to the formula $\exists x.\exists y.R(x,y)$.

(c) $\forall x.\forall y.\forall z.(x = z \to y = z \to x = y)$

Solution:

1.	x_0	fresh
2.	y_0	fresh
3.	z_0	fresh
4.	$x_0 = z_0$	assumption
5.	$y_0 = z_0$	assumption
6.	$y_0 = y_0$	=i
7.	$z_0 = y_0$	=e (5,6) with $\phi \equiv u = y_0$
8.	$x_0 = y_0$	=e (7,4) with $\phi \equiv x_0 = u$
9.	$y_0 = z_0 \to x_0 = y_0$	→i (5–8)
10.	$x_0 = z_0 \to y_0 = z_0 \to x_0 = y_0$	→i (4–9)
11.	$\forall z.(x_0 = z \to y_0 = z \to x_0 = y_0)$	∀i (3–10)
12.	$\forall y.\forall z.(x_0 = z \to y = z \to x_0 = y)$	∀i (2–11)
13.	$\forall x.\forall y.\forall z.(x = z \to y = z \to x = y)$	∀i (1–12)

4. (2x3.5pts) Some people claim that the sentence "*nobody trusts a politician*" can be formalised as $\forall x.\forall y.(P(x) \to \neg T(y,x))$ while others say it can be formalised as $\neg(\exists x.\exists y.(P(x) \land T(y,x)))$. Show that actually both formalisations are (provably) equivalent by giving a natural deducation proof of

$$\forall x.\forall y.(P(x) \to \neg T(y,x)) \dashv\vdash \neg(\exists x.\exists y.(P(x) \land T(y,x)))$$

Solution:

1.	$\forall x.\forall y.(P(x) \to \neg T(y,x))$	premise
2.	$\exists x.\exists y.(P(x) \wedge T(y,x))$	assumption
3.	x_0	fresh
4.	$\exists y.(P(x_0) \wedge T(y,x_0))$	assumption
5.	y_0	fresh
6.	$P(x_0) \wedge T(y_0,x_0))$	assumption
7.	$P(x_0)$	$\wedge$e$_1$ 6
8.	$T(y_0,x_0)$	$\wedge$e$_2$ 6
9.	$\forall y.(P(x_0) \to \neg T(y,x_0))$	$\forall$e 1 with x_0
10.	$P(x_0) \to \neg T(y_0,x_0)$	$\forall$e 9 with y_0
11.	$\neg T(y_0,x_0)$	$\to$e (10,7)
12.	$\bot$	$\neg$e (11,8)
13.	$\bot$	$\exists$e (4,5–12)
14.	$\bot$	$\exists$e (2,3–13)
15.	$\neg(\exists x.\exists y.(P(x) \wedge T(y,x)))$	$\neg$i (2–14)

1.	$\neg(\exists x.\exists y.(P(x) \wedge T(y,x)))$	premise
2.	x_0	fresh
3.	y_0	fresh
4.	$P(x_0)$	assumption
5.	$T(y_0,x_0)$	assumption
6.	$P(x_0) \wedge T(y_0,x_0)$	$\wedge$i (4,5)
7.	$\exists y.P(x_0) \wedge T(y,x_0)$	$\exists$i 6
8.	$\exists x.\exists y.P(x) \wedge T(y,x)$	$\exists$i 7
9.	$\bot$	$\neg$e
10.	$\neg T(y_0,x_0)$	$\neg$i (5–9)
11.	$P(x_0) \to \neg T(y_0,x_0)$	$\to$i (4–10)
12.	$\forall y.(P(x_0) \to \neg T(y,x_0))$	$\forall$i (3–11)
13.	$\forall x.\forall y.(P(x) \to \neg T(y,x))$	$\forall$i (2–12)

5. Consider the following semantic entailments

   (i) $\exists x.\forall y.P(x,y) \models \forall y.\exists x.P(x,y)$

   (ii) $\forall y.\exists x.P(x,y) \models \exists x.\forall y.P(x,y)$

(a) (1pt) What is a model for the language?

Solution:
A model $\mathcal{M}$ for the language consists in a domain $\mathcal{A} \neq \emptyset$, and a binary relation $P^{\mathcal{M}} \subseteq \mathcal{A} \times \mathcal{A}$.

(b) (2x3.5pts) Explain semantically (that is, reasoning with models) whether these entailments hold or not.

Solution:

(i) This entailment is valid.
Consider a model $\mathcal{M}$ such that $\mathcal{M} \models \exists x.\forall y.P(x,y)$. This tells us that for a certain $a \in \mathcal{A}$, $(a,b) \in P^{\mathcal{M}}$ for any $b \in \mathcal{A}$.
We need to show that $\mathcal{M} \models \forall y.\exists x.P(x,y)$. That is, $\mathcal{M} \models_{[y \mapsto b]} \exists x.P(x,y)$ for any $b \in \mathcal{A}$.
For this to hold, for any $b \in \mathcal{A}$ we need to find a $c \in \mathcal{A}$ such that $(c,b) \in P^{\mathcal{M}}$. But we already know that $(a,b) \in P^{\mathcal{M}}$ for any $b \in \mathcal{A}$, so our $c = a$ for any b.

(ii) This entailent is not valid.
Consider $\mathcal{A} = \mathbb{N}$ and $P^{\mathcal{M}} = \{(n,m) \in \mathbb{N}^2 \mid n = m^2\}$.
$\forall y.\exists x.P(x,y)$ is valid in this model since for any Natural number m, m^2 is also a Natural number.
On the other hand, $\exists x.\forall y.P(x,y)$ is not valid since it would mean that there is a Natural number n which is equal to the square of any Natural number m, which we know cannot be the case.

6. (3.5pts) We fix a language with a relation symbol R. Give a model which validates all the following formulae:

$$\forall x\ \neg R(x,x) \qquad \forall x \forall y \forall z\ (R(x,y) \wedge R(y,z) \rightarrow R(x,z))$$

$$\forall x\ \exists y\ R(x,y) \qquad \forall x \forall y\ (R(x,y) \rightarrow \exists z \exists t\ (R(x,z) \wedge R(z,t) \wedge R(t,y)))$$

Solution:
A model is given by the set of rational numbers with the $<$ relation.

7. (2x3.5pts) Explain whether the following LTL formulae are valid or not:

(a) $(FG(p) \wedge FG(q)) \rightarrow FG(p \wedge q)$

(b) $G(p \to Xp) \to (Gp \vee G(\neg p))$

Solution:

(a) The formula is valid.
If we have $FG(p)$ on a given path π, then then exists m such that $\pi, l \models p$ if $l \geq m$. If we also have $FG(q)$ then we have n such that $\pi, l \models q$ if $l \geq n$. Then we have $\pi, l \models p \wedge q$ for $l \geq N$ with $N = max(m, n)$.

(b) The formula is *not* valid.
Consider the path π where p is false at $\pi(0)$ and true at all $\pi(n)$ for $n > 0$. Then we have $\pi \models G(p \to Xp)$ but we don't have $\pi \models Gp$ and we don't have $\pi \models G(\neg p)$.

8. (a) (2pts) What is a model of CTL (Computation Tree Logic)?
   (b) (3+3.5pts) Explain whether the following CTL formulae are valid or not:
      (i) $EF(p \vee q) \to (EF(p) \vee EF(q))$
      (ii) $AF(p \vee q) \to AF(p) \vee AF(q)$

Solution:

(a) A model is given by a transition system $S, \to$ with no deadlock $N(s) = \{s' \mid s \to s'\} \neq \emptyset$ for s in S, and a labelling function $L(s) \subseteq P$, where P is the set of atoms.

(b) (i) The formula is valid.
If we have $s \models EF(p \vee q)$ we have a path $s = s_0 \to s_1 \dots$ starting from s with $s_n \models p \vee q$ for some n. We then have $s_n \models p$ or $s_n \models q$ and hence $s \models EF(p)$ or $s \models EF(q)$.

(ii) The formula is *not* valid.
Consider the model with 3 states s_0, s_1, s_2 and transitions $s_0 \to s_1,\ s_1 \to s_1,\ s_0 \to s_2,\ s_2 \to s_2$ and $L(s_0) = \emptyset,\ L(s_1) = p,\ L(s_2) = q$. We have $s_0 \models AF(p \vee q)$ but $AF(p)$ is not valid at s_0 and $AF(q)$ is not valid at s_0.

9. We consider a language with a predicate symbol $P(x)$, a unary function symbol s and a constant a. We define the term $t(n)$ by $t(0) = a$ and $t(n+1) = s(t(n))$. So, $t(1) = s(a),\ t(2) = s(s(a)), \dots$ Consider the formulae

$$\psi_1 = P(s(a)) \qquad \psi_2 = \forall x\ (P(x) \to P(s(x)))$$

(a) (1pt) For which n (or n's) do we have $\psi_1, \psi_2 \vdash P(t(n))$?

(b) (3.5pts) Explain your answer.

Solution:
This is the case exactly when $n > 0$. By induction, we show that if $n > 0$ then $\psi_1, \psi_2 \vdash P(t(n))$. Conversely, we can consider the model M, where the universe A is the set of natural numbers, and P^M is the set of numbers > 0. By *soundness*, if we have $\psi_1, \psi_2 \vdash P(t(n))$, we should have $t(n)^M > 0$ and hence $n = t(n)^M > 0$.

10. (3pts) Let $Pow(A)$ be the set of subsets of a given finite set A and $F : Pow(A) \rightarrow Pow(A)$ be a function such that $F(Y) \subseteq F(X)$ whenever $X \subseteq Y$ (for instance $F(X)$ can be $A - X$). Explain why there exists a subset Z of A such that $F(F(Z)) = Z$.

Solution:
We have that $Z \mapsto F(F(Z))$ is *monotone*. Hence it has a fixpoint.