# Logic in Computer Science

DAT060/DIT201 (7.5 hec)

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Tuesday 17th of August 2020

| Total: 60 points | |
|---|---|
| CTH: $\geqslant 30$: 3, $\geqslant 41$: 4, $\geqslant 51$: 5 | GU: $\geqslant$30: G, $\geqslant$46: VG |

No help material but dictionaries to/from English.

Write in English and as readable as possible (what we cannot read we cannot correct!!).

Write each new exercise in a new page!

**OBS:** All answers should be *carefully* motivated. Points will be deduced when you do not properly justify your answer.

**Good luck!**

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1. Give proofs in natural deduction of the following sequents:
   (a) (3 pts) $(q \to r) \wedge (q \vee p) \vdash (p \to q) \to (r \wedge q)$
   (b) (4 pts) $(p \to q) \to q \vdash \neg q \to p$
2. (a) (1pt) Without using truth tables, give a valuation for which the formula

$$(p \wedge q \to r \wedge s) \vee \neg(p \wedge s \to r)$$

   is false.
   (b) (2pts) Explain how you arrived to this valuation.
3. For each of the sequents below, prove using natural deduction that they are valid, or give a counter-model showing that they are not.
   (a) (3pts) $\forall x \forall y (R(x,y) \to R(y,x)), \forall x \forall y \neg(R(x,y) \wedge R(y,x)) \vdash \forall x \forall y \neg R(x,y)$
   (b) (3pts) $\forall x \forall y (R(x,y) \to R(y,x)), \forall x \forall y \forall z (R(x,y) \wedge R(y,z) \to R(x,z)) \vdash \forall x R(x,x)$
   (c) (3.5pts) $\forall x (P(x,x) \vee \forall y Q(x,y)) \vdash \forall x (\exists y P(x,y) \vee Q(x,x))$
   (d) (3.5pts) $\forall x (x = a \vee x = b), \exists x P(x) \vdash \neg P(a) \to P(b)$
4. Consider the following semantic entailments
   **i)** $\forall x \neg \forall y (P(x,y) \to Q(x,y)) \models \forall x \exists y P(x,y)$
   **ii)** $\forall x (P(x,x) \to \forall y Q(x,y)) \models \forall x \neg P(x,x) \vee \forall x \forall y Q(x,x)$

**iii)** $\exists x(P(x,x) \land \forall y Q(x,y)) \models \exists x(\exists y P(x,y) \land Q(x,x))$

(a) (1 pts) What is a model for the language of these entailments?

(b) (3x2 pts) Explain semantically (that is, reasoning with models) whether these entailments are valid or not.

5. Show that the following entailment is *not* valid (3 pts)

$$\forall x\,(f(f(x)) = x) \models \forall x\,(f(x) = x)$$

6. Explain why the following LTL formula is *not* valid (3 pts)

$$(Fp \land G(p \to XFp)) \to FGp$$

7. Consider the language with one function symbol $f$ and one constant $a$. We have a model $\mathcal{M}$ for the language consisting of the set of natural numbers $\mathbb{N}$, and interpretations $a^{\mathcal{M}} = 0$ and $f^{\mathcal{M}}(n) = n + 1$. Give an example of a formula which holds in this model (2 pts) and an example of a formula which does not hold in this model (2 pts).

8. Explain when a LTL formula is *valid* on a given model (3 pts). Explain then why if the formula $G(p \to q)$ is valid on a given model, then so is the formula $G(p) \to G(q)$ (3 pts).

9. Given an example of a LTL/CTL model for which the CTL formula $AG(EF\,p)$ is *valid* but for which the LTL formula $G(F\,p)$ is *not valid* (4 pts).

10. Let $S$ be a set and $A, B$ two given subsets of $S$. Let $F : Pow(S) \to Pow(S)$ be the function $F(X) = (X \cap A) \cup B$. Explain why $F$ is *monotone* (2 pts). What is the *least* fixpoint (2 pts) of $F$ and what is the *greatest* fixpoint (2 pts) of $F$?

11. Explain why the following LTL formula is *valid* (4 pts)

$$(Fp \land Fq) \to (F(p \land Fq) \lor F(q \land Fp))$$