DATAVETENSKAP
Göteborgs universitet

Tentamensskrivning i **Logic in Computer Science**
2007-04-10 EM.
(DIT200/DAT060) 5p
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All answers and solutions must be carefully motivated!

1. Let ψ be the formula

   $[\forall x \exists y\ R(x,y)] \rightarrow \exists y\ R(y,y)$

   a) Explain why we don't have $\vdash \psi$ (2p)

   b) Give a model M for which we have $M \models \psi$ (1p)

   Solution: By *soundness*, it is enough to show to build a model in which ψ is not valid. We can take $M = \mathbb{N}$ and interpet $R(x,y)$ by $x < y$. Then $\forall x \exists y\ R(x,y)$ is valid in this model but not $\exists y\ R(y,y)$.

   A model which validates ψ is obtained by changing the interpretation $R(x,y)$ to $x = y$.

2. We consider the language with two unary predicate symbol P, Q and a model $M = A,\ P^M \subseteq A,\ Q^M \subseteq A$. Assuming $M \models \forall x\ (P(x) \vee Q(x))$, show that we must have

   $$M \models \forall x\ P(x) \vee \exists x\ Q(x)$$

   by looking at the two cases: $P^M = A$ or $P^M \neq A$ (1p).

   Explain then why we can conclude from this that we have (1p)

   $$\forall x\ (P(x) \vee Q(x)) \vdash (\forall x\ P(x)) \vee (\exists x\ Q(x))$$

   Solution: If we have $P^M = A$ then we have $\forall x\ P(x)$ valid in M and so $M \models \forall x\ P(x) \vee \exists x\ Q(x)$. Otherwise $P^M \neq A$. Since we have $M \models \forall x\ (P(x) \vee Q(x))$ we have $A = P^M \cup Q^M$ and so $P^M \neq A$ implies $Q^M \neq \emptyset$. This implies that $\exists x\ Q(x)$ is valid in M and so $M \models \forall x\ P(x) \vee \exists x\ Q(x)$ holds in this case as well.

   By *completeness* we deduce that we have

   $$\forall x\ (P(x) \vee Q(x)) \vdash (\forall x\ P(x)) \vee (\exists x\ Q(x))$$

3. Compute a CNF for the formula

   $(p \wedge q) \vee (q \wedge r) \vee (r \wedge p)$ (1.5p)

   $\neg(r \vee (\neg p \rightarrow \neg q))$ (1.5p)

   Solution: See textbook, Chapter 1.5

4. Give a proof in natural deduction of

   a) $(\neg p \wedge \neg q) \rightarrow \neg(p \vee q)$ (1.5p)

   b) $((\forall x\ P(x)) \vee (\forall x\ Q(x))) \rightarrow \forall x\ (P(x) \vee Q(x))$ (2p)

   c) $\forall x \forall y \forall z\ [(R(x,y) \wedge R(z,y)) \rightarrow R(x,z)], \forall x\ R(x,x) \vdash \forall x \forall y\ [R(x,y) \rightarrow R(y,x)]$ (2p)

Solution: See textbook.

5. Say whether these consequences are true or false (if it is true give a proof, and if it is false give a model where it is not valid):

   a) $\exists x\ [(P(x) \to Q(x)) \wedge (Q(x) \to P(x))], \forall x\ P(x) \vdash \forall x\ Q(x)$ (2p)

   b) $\forall x \forall y \forall z\ [(R(x,y) \wedge R(z,y)) \to R(x,z)] \vdash \forall x \forall y\ [R(x,y) \to R(y,x)]$ (1.5p)

   c) $\forall x \exists y \forall z\ P(x,y,z) \vdash \forall x \forall z \exists y\ P(x,y,z)$ (1p)

   Solution: The first formula is not valid. We can take $A = \{0,1\}$ and $P^M = \{0,1\}$ and $Q^M = \{0\}$.

   The second formula is not valid. We can take $A = \{0,1\}$ and $R^M = \{(0,1),(0,0)\}$.

   The last formula is valid. One way to say it is to notice that we have $A(x) \vdash B(x)$ where $A(x)$ is $\exists y \forall z\ P(x,y,z)$ and $B(x)$ is $\forall z \exists y\ P(x,y,z)$. Hence $\forall x.A(x) \vdash \forall x.B(x)$ is valid as well.

6. We consider the following CTL model $(S, \to, L)$ where S has 4 states s_0, s_1, s_2, s_3 and we have $s_0 \to s_1, s_0 \to s_3,\ s_1 \to s_1,\ s_1 \to s_2,\ s_2 \to s_0,\ s_2 \to s_3,\ s_3 \to s_0$ and $L(s_0) = \{a\},\ L(s_1) = \{c\},\ L(s_2) = \{b\}$ and $L(s_3) = \{b,c\}$. What are the states s for which we have?

   a) $s \models EX\ (EX\ c)$ (1.5p)?

   b) $s \models AG\ (EF\ b)$ (1.5p)

   Solution: In both cases the formulae are valid for *all* states.

7. Is the following LTL formula

$$G\ (p \vee q) \to (FG\ p \vee GF\ q)$$

   valid? Explain why (2p)

   Solution: This formula is valid. Let σ be a trace. We have at all times n that $\sigma^n \models p$ or $\sigma^n \models q$. If $GF\ q$ is not valid this means that there is N such that $\sigma^n \models \neg q$ for all $n \geqslant N$. But we have then $\sigma^n \models p$ for all $n \geqslant N$ and hence $\sigma \models FG\ p$.

8. Given the atomic formulae $p_1, \ldots, p_n$ we define the formulae $C = p_1 \wedge \ldots \wedge p_n$ and $D = p_1 \vee \ldots \vee p_n$. Explain why the following equivalence is valid (2p)

$$(D \to C) \quad \leftrightarrow \quad (p_1 \to p_2) \wedge (p_2 \to p_3) \wedge \ldots \wedge (p_{n-1} \to p_n) \wedge (p_n \to p_1)$$

   Solution: Write $p_{n+1} = p_1, p_{n+2} = p_2, \ldots$

   Assume $D \to C$. Since $p_i \to D$ and $C \to p_{i+1}$ hold we have then $p_i \to p_{i+1}$ for all i.

   Assume that we have $p_i \to p_{i+1}$ for all i. Then by induction we have $p_i \to p_{i+k}$ for all k and hence $p_i \to p_j$ for all i, j. This shows that we have $p_i \to C$ for all i and hence $D \to C$.